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Analyse Harmonique des Formes Différentielles sur l'Espace Hyperbolique Réel

THÈSE

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par

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À Gelindo et Teresina

Ceux qui ont commencé à apprendre enchaînent les formules, mais n'en savent pas encore le sens ; car il faut qu'elles mûrissent avec nous. Or c'est là une chose qui demande du temps.

ARISTOTE, *Éthique à Nicomaque.*

« Prendre son vol » chaque jour ! Au moins un moment qui peut être bref, pourvu qu'il soit intense. Chaque jour un « exercice spirituel » — seul ou en compagnie d'un homme qui lui aussi veut s'améliorer.
Exercices spirituels. Sortir de la durée. S'efforcer de dépouiller tes propres passions, les vanités [...] Dépouiller la pitié et la haine. Aimer tous les hommes libres. S'éterniser en se dépassant.

Georges FRIEDMANN, *La Puissance et la Sagesse* (1970).

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Introduction

Un peu d'histoire

L'analyse harmonique est née au milieu du dix-huitième siècle du désir de résoudre l'équation aux dérivées partielles

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

fournie par le problème de « la corde vibrante ». Des mathématiciens tels que d'Alembert, Euler, Bernoulli et Parseval découvrirent peu à peu que la solution d'une telle équation différentielle pouvait s'exprimer à l'aide de séries trigonométriques. Il fallut cependant attendre le début du siècle suivant pour qu'avec Fourier et ses travaux sur la diffusion de la chaleur soit correctement formalisé le principe de la décomposition d'une fonction en série ou intégrale de fonctions trigonométriques « élémentaires ». Rappelons qu'une fonction convenable sur l'intervalle $[-\pi, \pi]$ se décompose en séries de Fourier comme

$$f(x) = \sum_{n\in\mathbb{Z}} e^{inx}\widehat{f}(n), \tag{0.1}$$

où $\widehat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} dx\, e^{-inx} f(x)$, et qu'une fonction définie sur la droite réelle se décompose sous certaines conditions comme

$$f(x) = \int_{\mathbb{R}} dy\, e^{2i\pi xy}\widehat{f}(y), \tag{0.2}$$

où la fonction $y \mapsto \widehat{f}(y) = \int_{\mathbb{R}} dx\, e^{-2i\pi xy} f(x)$ est la *transformation de Fourier* de f. Ce furent les premiers exemples d'une théorie qui n'allait cesser de se développer.

Il est désormais bien connu que sous le principe de décomposition introduit par Fourier se cache l'action d'un groupe G — respectivement le cercle $\mathbb{R}/2\pi\mathbb{Z}$ et la droite $\mathbb{R}$ dans les exemples ci-dessus — sur l'espace $L^2(G)$ des fonctions de carré intégrable sur G. Dans ce sens, les fonctions $x \mapsto e^{inx}$ $(n \in \mathbb{Z})$ et $x \mapsto e^{2i\pi xy}$ $(y \in \mathbb{C})$ peuvent être vues commes les *caractères* respectifs des groupes abéliens $\mathbb{R}/2\pi\mathbb{Z}$ et $\mathbb{R}$, c'est-à-dire les homomorphismes (continus) de ces groupes dans le corps multiplicatif $\mathbb{C}^*$.

Ainsi les décompositions (0.1) et (0.2) se généralisent-elles à tous les groupes abéliens finis ou localement compacts. Par exemple, si G est un groupe abélien localement compact, et si $\widehat{G}$ désigne l'ensemble des caractères *unitaires* (i.e. de norme 1) de G [1], la transformation de Fourier d'une fonction f sur G se définit par

$$\widehat{f}(\chi) = \int_G dx\, \overline{\chi(x)} f(x) \tag{0.3}$$

pour tout $\chi \in \widehat{G}$, dx étant une mesure de Haar normalisée sur G. Weil a montré en 1940 qu'il existe une mesure de Haar $d\nu(\chi)$ sur $\widehat{G}$, appelée couramment *mesure de Plancherel*, telle que pour toute $f \in C(G) \cap L^1(G)$ vérifiant $\widehat{f} \in L^1(\widehat{G})$,

$$f(x) = \int_{\widehat{G}} d\nu(\chi)\, \chi(x) \widehat{f}(\chi). \tag{0.4}$$

La *formule de Plancherel(-Parseval)* $\|f\|^2_{L^2(G)} = \|\widehat{f}\|^2_{L^2(\widehat{G})}$ en découle, et ainsi la transformation de Fourier s'étend en une bijection de $L^2(G; dx)$ sur $L^2(\widehat{G}; d\nu(\chi))$.

Lorsque le groupe G n'est plus commutatif, il devient nécessaire de considérer des généralisations pluridimensionnelles des caractères : les *représentations* du groupe. Une représentation (continue) π d'un groupe localement compact G sur un espace de Hilbert $\mathcal{H}$ est un homomorphisme (continu) de G dans le groupe $GL(\mathcal{H})$. Ainsi les caractères précédemment introduits sont-ils des représentations de dimension un, la dimension d'une représentation étant par définition la dimension (complexe) de l'espace de Hilbert qui lui est associé. Une représentation $(\pi, \mathcal{H})$ sera dite *unitaire* si chaque $\pi(x)$ est un endomorphisme unitaire, et *irréductible* si $\mathcal{H}$ ne contient pas de sous-espaces $\pi(G)$-invariants non triviaux. Mautner et Segal ont démontré (indépendamment) en 1950 la formule de Plancherel pour la classe relativement générale des groupes localement compacts « unimodulaires » et « de type I » (nous ne désirons pas expliciter ici ces notions). Notamment, la formule d'inversion (0.4) reste valable, sachant que $\widehat{f}(\pi)$ est une matrice et que $\widehat{G}$ désigne alors l'ensemble des (classes d'équivalence de) représentations unitaires continues irréductibles de G.

Cependant, cette belle théorie reste abstraite dans la mesure où elle ne décrit ni la partie du dual unitaire $\widehat{G}$ (si toutefois $\widehat{G}$ est lui-même connu) intervenant dans (0.4),

[1] En réalité, $\widehat{G}$ est également un groupe abélien localement compact (appelé *dual unitaire* de G).

ni la mesure de Plancherel $d\nu(\pi)$. Il a fallu près de vingt-cinq ans à Harish-Chandra pour parvenir à les rendre explicites dans le cas des groupes *de Lie* (i.e. munis d'une structure de variété différentiable) (essentiellement) semi-simples, ses efforts culminant dans les articles mémorables [HC75, HC76a, HC76b]. Ce type de groupe entre dans la catégorie étudiée par Mautner et Segal, et de nombreux groupes de matrices en fournissent des exemples classiques, comme les groupes spéciaux linéaires, certains groupes d'isométries associés à une forme quadratique, les groupes symplectiques, etc.

Nous n'insistons pas davantage sur la théorie générale de l'analyse harmonique et ses liens avec la théorie des représentations. Pour un historique plus détaillé et de nombreuses références, voir le très bel exposé [Kna96a, Kna96b]. Nous allons maintenant nous concentrer sur le cadre semi-simple, et plus exactement sur l'un de ses aspects particuliers.

L'analyse harmonique des fonctions sur les espaces hyperboliques [2]

Supposons dorénavant que G est un groupe de Lie réel semi-simple, connexe, non compact et de centre fini, et soit K un sous-groupe compact maximal de G. Il est bien connu qu'il existe alors un sous-groupe abélien $A \simeq \mathbb{R}^r$ de G tel que $G \simeq KAK$: c'est la *décomposition de Cartan* de G, et l'entier r est appelé *rang réel* de G. De plus, le quotient G/K est une variété riemannienne, symétrique (en un certain sens géométrique), de *type non compact* (i.e. de courbure de Ricci négative) et de *rang* r (ce qui signifie que r est la dimension maximale des sous-variétés plates totalement géodésiques de G/K). La décomposition de l'espace $L^2(G/K)$ résulte alors de celle de $L^2(G)$ établie par Harish-Chandra, en remarquant simplement qu'une fonction sur G/K s'identifie à une fonction K-invariante à droite sur G [3]. Introduisons quelques notations afin d'être plus explicite.

Pour simplifier, plaçons-nous tout de suite dans le cadre auquel nous nous restreindrons presque toujours dans notre travail de thèse, celui où G/K est de rang un. Ainsi $A \simeq \{a_t,\ t \in \mathbb{R}\}$, et G/K est nécessairement l'un des quatre *espaces hyperboliques* $H^n(\mathbb{R})$, $H^n(\mathbb{C})$, $H^n(\mathbb{H})$, $H^2(\mathbb{O})$; par exemple, $H^n(\mathbb{R}) \simeq SO_e(n,1)/SO(n)$, $H^n(\mathbb{C}) \simeq SU(n,1)/S(U(n)\times U(1))$. Par ailleurs, nous savons que G est difféomorphe

[2] Les références classiques pour ce qui suit sont [Hel78, Hel84, Hel94], [Far82], [Koo84], [GV88].

[3] En fait, la formule de Plancherel sur $L^2(G/K)$ précéda la formule de Plancherel sur $L^2(G)$ d'une dizaine d'années.

au produit KAN, où N est un sous-groupe nilpotent (ou abélien dans le cas réel) de G (c'est une *décomposition d'Iwasawa* de G). Notons H la projection de $G = KAN$ sur A définie par $H(g) = t$ si $g = ka_tn$, $\underline{k}$ la projection de $G = KAN$ sur K, $\mathfrak{g}$ et $\mathfrak{a} \simeq \mathbb{R}$ les algèbres de Lie respectives de G et A, $\mathfrak{g}_{\mathbb{C}}$ et $\mathfrak{a}_{\mathbb{C}}$ leurs complexifiées, et soit M le centralisateur de A dans K. Le quotient K/M s'identifie naturellement (comme variété) au *bord* de l'espace hyperbolique G/K, i.e. à la sphère unité dans $\mathbb{F}^n$ si $G/K = H^n(\mathbb{F})$.

Nous allons d'abord donner dans ce nouveau contexte l'analogue des caractères $x \mapsto e^{2i\pi xy}$ et $x \mapsto \chi(x)$ intervenant respectivement dans (0.2) et (0.4). Nous avons mentionné plus haut que la généralisation de la notion de caractère (unitaire) au cadre non abélien était celle de représentation (unitaire, irréductible). Cependant, seules certaines représentations de G vont intervenir dans la définition de la transformation de Fourier sur G/K et jouer un rôle pour la formule de Plancherel. Voyons lesquelles. Soit $P = MAN$ (P est un sous-groupe, dit *parabolique*, de G), et pour $\lambda \in \mathfrak{a}_{\mathbb{C}}^* \simeq \mathbb{C}$, définissons un caractère χ_λ de P par

$$\chi_\lambda(ma_tn) = e^{i\lambda t}.$$

On obtient alors une représentation π_λ de G en « induisant » le caractère χ_λ de P au groupe tout entier. Sans entrer dans les détails d'une telle construction, disons que π_λ agit sur l'espace $L^2(K/M)$ par :

$$\{\pi_\lambda(x)f\}(kM) = e^{-(i\lambda+\rho)H(x^{-1}k)} f(\underline{k}(x^{-1}k)M),$$

où $\rho = \frac{d}{2}(n+1) - 1$ si $G/K = H^n(\mathbb{F})$ et $d = \dim_{\mathbb{R}} \mathbb{F}$ (en réalité, ρ a une signification algébrique). Les représentations π_λ sont appelées *séries principales sphériques* de G ; elles sont unitaires si λ est réel, et génériquement irréductibles.

Nous pouvons maintenant définir la transformation de Fourier d'une fonction f sur G/K : pour $kM \in K/M$, $\lambda \in \mathbb{C}$, on pose

$$\begin{aligned}\mathcal{H}(f)(\lambda, kM) &= \int_G dx\, \{\pi_\lambda(x)\mathbf{1}_{K/M}\}(kM)\, f(x) \\ &= \int_G dx\, e^{-(i\lambda+\rho)H(x^{-1}k)} f(x), \end{aligned} \tag{0.5}$$

où dx est une mesure de Haar sur G convenablement normalisée (mais l'intégration peut aussi se faire sur G/K). Remarquons que $kM \mapsto \mathcal{H}(f)(\lambda, kM) \in C^\infty(K/M)$ si par exemple on suppose $f \in C_c^\infty(G/K)$. Le théorème de Plancherel s'énonce alors de la manière suivante.

Théorème 0.1. *Normalisons la mesure $d(kM)$ sur K/M de telle sorte que le volume total soit égal à 1. Alors :*

(i) *Il existe une mesure $d\nu(\lambda)$ sur $\mathbb{R}$ telle qu'on ait la formule d'inversion*

$$f(x) = \int_{K/M} d(kM) \int_0^\infty d\nu(\lambda) e^{(i\lambda-\rho)H(x^{-1}k)} \mathcal{H}(f)(\lambda, kM) \qquad (0.6)$$

pour toute fonction $f \in C_c^\infty(G/K)$.

(ii) *La transformation de Fourier $\mathcal{H}$ s'étend en une isométrie bijective de $L^2(G/K)$ sur $L^2([0,\infty[\times K/M; d\nu(\lambda)d(kM))$.*

La mesure de Plancherel $d\nu(\lambda)$ est bien connue : elle s'exprime sous la forme

$$d\nu(\lambda) = \text{cst } d\lambda \, |c(\lambda)|^{-2},$$

où c est la *fonction de Harish-Chandra* explicitement donnée par la formule de Gindikin-Karpelevič en termes de fonctions Γ d'Euler :

$$c(\lambda) = \frac{\Gamma(dn-1)}{\Gamma(\frac{dn-1}{2})} \frac{\Gamma(i\lambda)}{\Gamma(i\lambda + \frac{d(n-1)}{2})} \frac{\Gamma(\frac{i\lambda}{2} + \frac{d(n-1)}{4})}{\Gamma(\frac{i\lambda}{2} + \frac{d(n-1)}{4} + \frac{d-1}{2})},$$

et où $d\lambda$ désigne la mesure de Lebesgue sur $\mathbb{R}$.
Diverses méthodes permettent d'aboutir à la démonstration du Théorème 0.1. L'une d'elles consiste à se ramener à *l'analyse sphérique sur G/K*, c'est-à-dire à l'analyse de Fourier des fonctions *radiales* (ou *bi-K-invariantes*) sur G. Cette démarche présente en outre l'intérêt d'avoir de nombreuses applications, telles que la résolution d'équations aux dérivées partielles classiques (équations des ondes, de la chaleur, etc.) ou la « formule des traces » de Selberg. Pour finir ce paragraphe, nous en donnons les principaux résultats.

Considérons donc dorénavant les fonctions bi-K-invariantes sur G, i.e. telles que

$$f(k_1 x k_2) = f(x) \qquad (\forall x \in G, \forall k_1, k_2 \in K).$$

De telles fonctions sont également appelées radiales, puisque la décomposition de Cartan $G = KAK$ permet de se restreindre à des calculs sur $A \simeq \mathbb{R}$. Si $F(G)$ est un espace fonctionnel sur G, nous noterons $F(G)^\natural$ le sous-espace constitué des fonctions bi-K-invariantes.

En premier lieu, nous devons donner l'analogue des « noyaux de Poisson » $x \mapsto e^{-(i\lambda+\rho)H(x^{-1}k)}$ intervenant dans la définition de la transformation de Fourier (0.5). Ce seront des coefficients matriciels des séries principales sphériques π_λ introduites précédemment : pour $\lambda \in \mathbb{C}$ et $x \in G$, on pose

$$\begin{aligned}\varphi_\lambda(x) &= \big(\pi_\lambda(x)\mathbf{1}, \mathbf{1}\big)_{L^2(K/M)} \\ &= \int_{K/M} d(kM) e^{-(i\lambda+\rho)H(x^{-1}k)} \\ &= \int_K dk\, e^{-(i\lambda+\rho)H(x^{-1}k)},\end{aligned} \tag{0.7}$$

les mesures de Haar sur les groupes compacts K et M étant normalisées de telle sorte que le volume total soit égal à 1. Les fonctions φ_λ sont appelées *fonctions sphériques* sur G/K. Il est facile de remarquer qu'elles vérifient l'équation différentielle

$$\Delta\varphi_\lambda = -(\lambda^2 + \rho^2)\varphi_\lambda, \tag{0.8}$$

où Δ est le *Laplacien* des fonctions sur G/K [4]. De plus, les fonctions sphériques possèdent les propriétés suivantes :

(i) $\varphi_{-\lambda} = \varphi_\lambda$;
(ii) $\varphi_\lambda(x^{-1}) = \varphi_\lambda(x)$;
(iii) $\varphi_\lambda \in C^\infty(G)^\natural$;
(iv) $\varphi_{\pm\lambda}$ est l'unique solution normalisée ($\varphi_\lambda(e) = 1$) de (0.8) ;
(v) φ_λ est la *transformée de Poisson* $\mathcal{P}_\lambda(f)(xK) = (\pi_\lambda(x)\mathbf{1}, \bar{f})$ d'une fonction sur le bord K/M (la fonction $f \equiv 1$) ;
(vi) $\varphi_\lambda(a_t) = \phi_\lambda^{(\frac{dn}{2}-1, \frac{d}{2}-1)}(t)$, où $\phi_\lambda^{(\alpha,\beta)}$ désigne la fonction hypergéométrique *de Jacobi* d'indices α et β, et est définie comme l'unique solution analytique, paire et normalisée ($\phi(0) = 1$) de l'équation différentielle

$$\frac{d^2\phi}{dt^2} + \{(2\alpha+1)\coth t + (2\beta+1)\,\mathrm{th}\, t\}\frac{d\phi}{dt} + \{\lambda^2 + (\alpha+\beta+1)^2\}\phi = 0.$$

La transformation de Fourier *sphérique* d'une fonction $f \in C_c^\infty(G)^\natural$ se définit alors de la façon suivante : pour $\lambda \in \mathbb{C}$,

$$\mathcal{H}(f)(\lambda) = \int_G dx\, \varphi_\lambda(x^{-1}) f(x) = \int_G dx\, \varphi_\lambda(x) f(x) \quad \in \mathbb{C}. \tag{0.9}$$

[4] C'est un fait coutumier que d'étudier l'analyse harmonique à travers le spectre d'opérateurs différentiels.

Il est ici important de remarquer que, grâce à la radialité des fonctions considérées et à la propriété (vi) ci-dessus, l'analyse sphérique sur les espaces hyperboliques se ramène à l'analyse de Jacobi de fonctions d'une variable réelle, dont la théorie complète fut développée par Flensted-Jensen et Koornwinder dans les années 1970 (cf. [Koo84] pour un exposé d'ensemble). En quelque sorte, on peut « oublier » la structure algébrique sous-jacente pour se ramener à des manipulations analytiques qui présentent l'avantage d'être « transparentes ». Voici maintenant le théorème de Plancherel.

Théorème 0.2.

(i) *Avec les notations du Théorème 0.1, on a la formule d'inversion*

$$f(x) = \int_0^\infty d\nu(\lambda)\varphi_\lambda(x)\mathcal{H}(f)(\lambda) \qquad (\forall x \in G) \tag{0.10}$$

pour toute fonction $f \in C_c^\infty(G)^\natural$.

(ii) *La transformation de Fourier sphérique* $\mathcal{H}$ *s'étend en une isométrie bijective de* $L^2(G)^\natural$ *sur* $L^2([0,\infty[;d\nu(\lambda))$.

Nous voulons donner également le théorème de Paley-Wiener pour cette transformation. Pour $R > 0$, notons $C_R^\infty(G)^\natural$ le sous-espace de $C_c^\infty(G)^\natural$ dont les éléments sont à support inclus dans la boule fermée $\overline{B}(o,R)$ de G/K, et $PW_R(\mathbb{C})$ l'espace des fonctions h entières et paires sur $\mathbb{C}$, vérifiant de plus la condition

$$\forall N \in \mathbb{N}, \quad \sup_{\lambda\in\mathbb{C}} |h(\lambda)|(1+|\lambda|)^N e^{-R|\operatorname{Im}\lambda|} < +\infty.$$

Théorème 0.3. *La transformation de Fourier sphérique* $\mathcal{H}$ *est un isomorphisme de* $C_R^\infty(G)^\natural$ *sur* $PW_R(\mathbb{C})$.

Lorsqu'on s'intéresse à certaines E.D.P. classiques sur G/K, il est commode d'utiliser la factorisation

$$\mathcal{H} = \mathcal{F} \circ \mathcal{A},$$

où $\mathcal{F}$ est la transformation de Fourier euclidienne définie par

$$\mathcal{F}(f)(\lambda) = \int_{\mathbb{R}} dt\, e^{i\lambda t} f(t),$$

et $\mathcal{A}$ est la *transformation d'Abel* définie sur $C_c(G)^\natural$ par

$$\mathcal{A}(f)(t) = e^{-\rho t}\int_N dn\, f(na_t) = e^{\rho t}\int_N dn\, f(a_t n),$$

et dont on connaît (par différents travaux) une expression explicite.

Après ce rapide tour d'horizon de l'analyse harmonique sur les espaces hyperboliques, nous en venons maintenant au sujet de notre étude.

Présentation des résultats

Le travail réalisé dans cette thèse donne une généralisation des notions et des résultats principaux de l'analyse harmonique L^2 sur l'espace hyperbolique réel $H^n(\mathbb{R})$ lorsqu'on considère des formes différentielles plutôt que des fonctions. L'analyse de Fourier sur un fibré vectoriel homogène au-dessus de l'espace hyperbolique réel — ou plus généralement de n'importe quel espace symétrique riemannien de type non compact G/K — présente évidemment un intérêt en soi, mais l'étude particulière du fibré le plus naturel et le plus « simple », celui des formes différentielles, possède (au moins) deux avantages : l'un est de pouvoir exposer de manière assez concrète le contenu de cette étude, notamment pour le lecteur familiarisé avec le cas des fonctions, et l'autre réside dans les rapports étroits qu'entretient le sujet étudié avec d'autres domaines tels que la $(\mathfrak{g}, K)$-cohomologie, la géométrie riemannienne ou la physique mathématique.

De nombreux auteurs ont contribué à une meilleure connaissance de l'analyse harmonique sur des fibrés au-dessus d'espaces symétriques de type non compact G/K. La plupart des travaux explorent certains aspects particuliers de la théorie dans des cadres plus ou moins généraux, comme la transformation de Poisson et l'étude des opérateurs différentiels invariants ([Gai86, Gai88], [Olb94], [vdV94], [Yan94]), la théorie des fonctions sphériques ([War72], [Rad76], [Min92]), ou la résolution d'équations aux dérivées partielles classiques ([BOS94]). Seuls [Shi90, Shi94] et [Cam97a, Cam97b] exposent une théorie complète — mais nécessairement moins détaillée.

Dans le cas du fibré des formes différentielles sur l'espace hyperbolique réel, les progrès les plus remarquables concernent également la transformation de Poisson et/ou les opérateurs différentiels invariants ([Don81], [Gai86], [Str89], [vdV93], [Ale95], [CH96]), ainsi que le calcul de la mesure de Plancherel ([vdV93], [CH94]) — avec souvent, cependant, des restrictions importantes sur le degré des formes différentielles. Dans ce contexte également, l'unique étude globale est dûe à Badertscher et Reimann ([BR89]) mais elle ne concerne que les 1-formes (i.e. les champs de vecteurs) sur $H^n(\mathbb{R})$. C'est le désir de généraliser [BR89] à des formes de degré quelconque qui a en réalité motivé ce travail de thèse, et nous en venons maintenant à la présentation de nos résultats.

Nous conservons les mêmes notations et conventions que dans le paragraphe précédent, mais nous fixons dorénavant $G = SO_e(n,1)$, $K = SO(n)$ et donc $G/K = H^n(\mathbb{R})$, sauf mention contraire.
Nous commençons par remarquer que l'espace $\Gamma\,\Lambda^p H^n(\mathbb{R})$ des p-formes différentielles sur $H^n(\mathbb{R})$ ($0 \leqslant p \leqslant n$) s'identifie de manière naturelle et bien connue à l'espace $\Gamma(G, K, \tau_p)$ des fonctions $f : G \to \Lambda^p\mathbb{C}^n$ *de type* τ_p *(à droite)*, c'est-à-dire telles que

$$f(xk) = \tau_p(k^{-1})f(x) \qquad (\forall x \in G, \forall k \in K),$$

où $\tau_p = \Lambda^p \operatorname{Ad}^*$ désigne la représentation coadjointe de K sur $\Lambda^p(\mathfrak{g}_{\mathbb{C}}/\mathfrak{k}_{\mathbb{C}})^* \simeq \Lambda^p\mathbb{C}^n$ (notons que τ_p est unitaire et équivalente à la représentation standard de K sur $\Lambda^p\mathbb{C}^n$). Lorsque $p = 0$ (cas des fonctions, auquel nous ferons souvent référence par le terme *cas scalaire*), on retrouve simplement le fait que l'on considère des fonctions K-invariantes à droite sur G. Nous remplacerons « Γ » par les autres préfixes classiques « C », « C^∞ », « C_c », « L^2 », etc. lorsque nous considérerons des fonctions de type τ_p respectivement continues, C^∞, continues à support compact, de carré intégrable, etc. L'espace $L^2(G, K, \tau_p)$ est de Hilbert pour le produit scalaire hermitien

$$(f, g) = \int_G dx\, \big(f(x), g(x)\big)_{\Lambda^p\mathbb{C}^n} .$$

Enfin, une dernière remarque préliminaire est que l'opérateur $*$ de Hodge induit un isomorphisme $\Gamma\,\Lambda^p H^n(\mathbb{R}) \simeq \Gamma\,\Lambda^{n-p} H^n(\mathbb{R})$ (cf. par exemple [War83]), et, partant, qu'il nous suffira désormais de restreindre notre étude aux valeurs $p \leqslant \frac{n}{2}$. Par commodité, nous supposerons également $p \geqslant 1$ dans la suite de cet exposé.

Notre objectif principal est donc de décomposer l'espace $L^2(G, K, \tau_p)$ en composantes « irréductibles » sous l'action de G, et d'obtenir ainsi le théorème de Plancherel pour la transformation de Fourier sur cet espace. Nous y parviendrons de deux façons : d'une part, en particularisant au K-type τ_p le théorème de Plancherel dû à Harish-Chandra pour $L^2(G)$. C'est l'approche « abstraite » (algébrique) du problème, qui utilise essentiellement la théorie des représentations des groupes G et K. Elle a l'avantage d'être à la fois instructive et relativement rapide, mais l'inconvénient d'être (trop) peu explicite. La seconde façon de procéder consiste à développer d'abord une théorie analytique sphérique (i.e. déterminer l'analogue des fonctions sphériques φ_λ sur G/K, et énoncer le théorème de Plancherel pour la transformation de Fourier sphérique), d'où découleront ensuite les résultats escomptés. Nous verrons alors de manière frappante que les formules obtenues par cette approche sont aussi transparentes — quoiqu'un peu plus compliquées — que celles que nous avons rappelées dans le paragraphe précédent dans le cas des fonctions.

Commençons donc par exposer ce que nous avons appelé l'approche abstraite. Le point de départ est le théorème de Harish-Chandra (cf. par exemple [Wal92], Ch. 14)

$$L^2(G) \simeq \int_{\widehat{G}}^{\oplus} d\nu(\pi)\, \mathcal{H}_\pi \,\widehat{\otimes}\, \mathcal{H}_\pi^*, \tag{0.11}$$

qui donne la décomposition de $L^2(G)$ en intégrale directe d'espaces de Hilbert « élémentaires » sous l'action de la représentation birégulière B de G définie par

$$\{B(x,y)f\}(z) = f(x^{-1}zy).$$

L'équivalence (0.11) est réalisée *via* la transformation de Fourier

$$f \mapsto \widehat{f}(\pi) = \int_G dx\, \pi(x) f(x).$$

Si π_1, π_2 sont deux représentations de G, notons $\mathrm{Hom}_G(\mathcal{H}_{\pi_1}, \mathcal{H}_{\pi_2})$ l'espace des G-homomorphismes de $\mathcal{H}_{\pi_1}$ dans $\mathcal{H}_{\pi_2}$, c'est-à-dire des opérateurs $T : \mathcal{H}_{\pi_1} \to \mathcal{H}_{\pi_2}$ tels que

$$T \circ \pi_1(x) = \pi_2(x) \circ T \qquad (\forall x \in G)$$

(on dit alors que T entrelace π_1 avec π_2). En remarquant que $L^2(G,K,\tau_p)$ s'identifie à $\{L^2(G) \otimes \mathcal{H}_{\tau_p}\}^K$ (K agissant à droite sur $L^2(G)$), (0.11) implique

$$\begin{aligned} L^2(G,K,\tau_p) &\simeq \int_{\widehat{G}}^{\oplus} d\nu(\pi)\, \mathcal{H}_\pi \,\widehat{\otimes}\, \{\mathcal{H}_\pi^* \otimes \mathcal{H}_{\tau_p}\}^K \\ &\simeq \int_{\widehat{G}}^{\oplus} d\nu(\pi)\, \mathcal{H}_\pi \,\widehat{\otimes}\, \mathrm{Hom}_K(\mathcal{H}_\pi, \mathcal{H}_{\tau_p}). \end{aligned} \tag{0.12}$$

Grâce à cette écriture, nous voyons que seules les représentations $\pi \in \widehat{G}$ telles que $\mathrm{Hom}_K(\mathcal{H}_\pi, \mathcal{H}_{\tau_p}) \neq \{0\}$ interviendront effectivement dans la décomposition de

$L^2(G,K,\tau_p)$. Par ailleurs, les travaux de Harish-Chandra montrent que l'on peut se restreindre en rang un à ne considérer que deux types de représentations de G : les séries principales (minimales) et les séries discrètes. Rappelons quelques définitions et faits élémentaires à ce sujet.

Soit M le centralisateur de A dans K (ici, $M \simeq SO(n-1)$). Pour $\sigma \in \widehat{M}$ et $\lambda \in \mathbb{C}$, on définit une représentation du sous-groupe parabolique $P = MAN$ de G par

$$(\sigma \otimes e^{i\lambda} \otimes 1)(ma_t n) = e^{i\lambda t}\sigma(m),$$

et on pose $\pi_{\sigma,\lambda} = \mathrm{Ind}_P^G(\sigma \otimes e^{i\lambda} \otimes 1)$, ce qui fournit une représentation du groupe G sur l'espace

$$\begin{aligned}\mathcal{H}_{\sigma,\lambda} =& L^2(G,P,\sigma \otimes e^{i\lambda} \otimes 1)\\ :=& \{f : G \to \mathcal{H}_\sigma : f(xma_t n) = e^{-(i\lambda+\rho)t}\sigma(m)^{-1}f(x),\ f_{|K} \in L^2(K)\},\end{aligned}$$

(rappelons que $\rho = \frac{n-1}{2}$) l'action se faisant par translations à gauche :

$$\{\pi_{\sigma,\lambda}(x)f\}(y) = f(x^{-1}y).$$

Remarquons que l'opération de restriction $f \mapsto f_{|K}$ induit un isomorphisme de K-modules entre $\mathcal{H}_{\sigma,\lambda|K}$ et l'espace $L^2(K,M,\sigma)$ des fonctions L^2 de type σ sur K, sur lequel $\pi_{\sigma,\lambda}$ agit par

$$\{\pi_{\sigma,\lambda}(x)f\}(k) = e^{-(i\lambda+\rho)H(x^{-1}k)}\, f(\underline{k}(x^{-1}k)) \qquad (x \in G, k \in K).$$

Les représentations $\pi_{\sigma,\lambda}$, appelées *séries principales* de G, sont unitaires si $\lambda \in \mathbb{R}$, et pour $\sigma \in \widehat{M}$ fixée, irréductibles pour des valeurs « génériques » de λ.

Une représentation π de G sur $\mathcal{H}_\pi$ est une *série discrète* de G si l'une des conditions équivalentes suivantes est vérifiée :

(i) il existe un coefficient matriciel $\big(\pi(\cdot)u,v\big)_{\mathcal{H}_\pi}$ non nul de π dans $L^2(G)$;
(ii) tous les coefficients matriciels de π sont dans $L^2(G)$;
(iii) π se plonge dans la représentation (bi-)régulière de G sur $L^2(G)$.

Avec ces notations, la décomposition (0.12) devient

$$\begin{aligned}L^2(G,K,\tau_p) \simeq& \int_{W\backslash(\widehat{M}\times\mathfrak{a}^*)}^{\oplus} d\nu(\sigma,\lambda)\, \mathcal{H}_{\sigma,\lambda}\,\widehat{\otimes}\, \mathrm{Hom}_K(\mathcal{H}_{\sigma,\lambda},\mathcal{H}_{\tau_p})\\ &\oplus \sum\nolimits_{\widehat{G}_d}^{\oplus} d_\pi\, \mathcal{H}_\pi\,\widehat{\otimes}\,\mathrm{Hom}_K(\mathcal{H}_\pi,\mathcal{H}_{\tau_p}),\end{aligned} \qquad (0.13)$$

où W est le groupe de Weyl $W(\mathfrak{g},\mathfrak{a}) \simeq \{\pm 1\}$, $\widehat{G}_d$ la partie discrète de $\widehat{G}$, et d_π le poids de $\pi \in \widehat{G}_d$.
Désignons respectivement par $L^2(G,K,\tau_p)_c$ et $L^2(G,K,\tau_p)_d$ les parties continue et discrète dans le membre de droite de (0.13). La description exacte de $L^2(G,K,\tau_p)_c$ va s'obtenir grâce à la loi de réciprocité de Frobenius :

$$\mathrm{Hom}_K(\mathcal{H}_{\sigma,\lambda},\mathcal{H}_{\tau_p}) = \mathrm{Hom}_K(L^2(K,M,\sigma),\mathcal{H}_{\tau_p}) \simeq \mathrm{Hom}_M(\mathcal{H}_\sigma,\mathcal{H}_{\tau_p}).$$

Or nous avons les faits suivants :

(i) τ_p est irréductible si $p < \frac{n}{2}$ et $\tau_{\frac{n}{2}} = \tau^+_{\frac{n}{2}} \oplus \tau^-_{\frac{n}{2}}$, cette décomposition en facteurs irréductibles inéquivalents correspondant à la décomposition de $\mathcal{H}_{\tau_{\frac{n}{2}}}$ en sous-espaces propres pour l'opérateur de Hodge;

(ii) soit σ_p la représentation standard de $M \simeq SO(n-1)$ sur $\Lambda^p\mathbb{C}^{n-1}$. Alors, comme en (i), σ_p est (unitaire,) irréductible si $p < \frac{n-1}{2}$ et $\sigma_{\frac{n-1}{2}} = \sigma^+_{\frac{n-1}{2}} \oplus \sigma^-_{\frac{n-1}{2}}$;

(iii) la restriction de τ_p à M s'écrit

$$\tau_{p|_M} = \begin{cases} \sigma_p \oplus \sigma_{p-1} & \text{si } 1 \leqslant p < \frac{n-1}{2}, \\ \sigma_p^+ \oplus \sigma_p^- \oplus \sigma_{p-1} & \text{si } p = \frac{n-1}{2}, \\ 2\,\tau^\pm_{p}{}_{|_M} \sim 2\,\sigma_p \sim 2\,\sigma_{p-1} & \text{si } p = \frac{n}{2}. \end{cases}$$

Par conséquent, $\mathrm{Hom}_K(\mathcal{H}_{\sigma,\lambda},\mathcal{H}_{\tau_p}) \neq \{0\}$ si et seulement si $\sigma = \sigma_{p-1},\sigma_p$ ou $\sigma = \sigma_{p-1},\sigma_p^\pm$ (si $p = \frac{n-1}{2}$). De plus, quitte à considérer les sous-représentations $\tau^\pm_{\frac{n}{2}}$, cet espace est toujours de dimension un. Ainsi, lorsque par exemple $p < \frac{n-1}{2}$, en étudiant l'action du groupe de Weyl W sur les séries principales, on obtient

$$L^2(G,K,\tau_p)_c = \int_{\mathbb{R}_+}^{\oplus} d\nu_{p-1}(\lambda)\,\mathcal{H}_{\sigma_{p-1},\lambda} \,\oplus \int_{\mathbb{R}_+}^{\oplus} d\nu_p(\lambda)\,\mathcal{H}_{\sigma_p,\lambda}$$

et un résultat analogue dans les autres cas.
D'autre part, la description de $L^2(G,K,\tau_p)_d$ résulte en grande partie d'un calcul dû à Borel ([Bor85]), que nous reprenons et précisons dans l'Appendice A de notre travail :

Proposition 0.4.

$$L^2(G,K,\tau_p)_d = \begin{cases} \{0\} & si\ p \neq \frac{n}{2}, \\ \mathcal{H}_{\pi^+} \oplus \mathcal{H}_{\pi^-} & si\ p = \frac{n}{2}, \end{cases}$$

où π^+ et π^- sont les séries discrètes de G possédant un caractère infinitésimal trivial. De plus, $\mathcal{H}_{\pi^\pm}$ est exactement le sous-espace $L^2(G,K,\tau^\pm_{\frac{n}{2}})_\Delta$ constitué des fonctions de type $\tau^\pm_{\frac{n}{2}}$ sur G qui sont L^2 et harmoniques.

Nous en arrivons finalement au résultat suivant (Théorème 3.2).

Théorème 0.5 (Théorème de Plancherel abstrait pour $L^2(G, K, \tau_p)$). *L'espace $L^2(G, K, \tau_p)$ des p-formes différentielles L^2 sur l'espace hyperbolique réel $H^n(\mathbb{R})$ se décompose comme suit.*

- *Pour $p \neq \frac{n-1}{2}, \frac{n}{2}$:*

$$L^2(G, K, \tau_p) = \sum\nolimits^{\oplus}_{q=p-1,p} \int^{\oplus}_{\mathbb{R}_+} d\nu_q(\lambda)\, \mathcal{H}_{\sigma_q,\lambda}.$$

- *Pour $p = \frac{n-1}{2}$:*

$$L^2(G, K, \tau_p) = \int^{\oplus}_{\mathbb{R}_+} d\nu_{p-1}(\lambda)\, \mathcal{H}_{\sigma_{p-1},\lambda} \oplus \int^{\oplus}_{\mathbb{R}_+} d\nu_p^+(\lambda)\, \mathcal{H}_{\sigma_p^+,\lambda} \oplus \int^{\oplus}_{\mathbb{R}_+} d\nu_p^-(\lambda)\, \mathcal{H}_{\sigma_p^-,\lambda}.$$

- *Pour $p = \frac{n}{2}$:*

$$L^2(G, K, \tau_p) = \mathcal{H}_{\pi^+} \oplus \mathcal{H}_{\pi^-} \oplus 2 \int^{\oplus}_{\mathbb{R}_+} d\nu_q(\lambda)\, \mathcal{H}_{\sigma_q,\lambda} \qquad (q = p-1 \text{ ou } p),$$

où π^+ et π^- sont les séries discrètes de G possédant un caractère infinitésimal trivial et constituent la partie harmonique de $L^2(G, K, \tau_p^\pm)$.

La distinction de trois cas dans l'énoncé précédent reviendra sans cesse par la suite. Désormais, nous dirons que p est *générique* si $1 \leqslant p < \frac{n-1}{2}$, et *spécial* dans les cas $p = \frac{n-1}{2}, \frac{n}{2}$.

Nous consacrons la suite de cette introduction au développement de l'approche analytique « concrète » du théorème de Plancherel, par le biais de l'analyse sphérique. La notion de radialité (bi-K-invariance) des fonctions sur G correspondant au cas $p = 0$ est ici remplacée par celle de τ-radialité, pour $\tau \in \{\tau_1, \ldots, \tau_{\frac{n-1}{2}}, \tau_{\frac{n}{2}}^{\pm}\}$. *Fixons désormais τ dans cet ensemble.*
Nous dirons qu'une fonction $F : G \to \operatorname{End} \mathcal{H}_\tau$ est *τ-radiale* si elle vérifie

$$F(k_1 x k_2) = \tau(k_2)^{-1} F(x) \tau(k_1)^{-1} \qquad (x \in G,\ k_1, k_2 \in K).$$

Remarquons que si F est τ_p-radiale, pour tout $\xi \in \mathcal{H}_{\tau_p}$, la fonction $x \mapsto F(x)\xi$ est de type τ_p sur G et s'identifie donc à une p-forme sur G/K. L'espace des fonctions τ-radiales sur G sera noté $\Gamma(G, K, \tau, \tau)$, étant entendu que nous remplacerons « Γ » par

« C », « C_c », « C^∞ », etc. si nécessaire. Le produit scalaire hermitien sur $L^2(G,K,\tau,\tau)$ est donné par

$$(F,H) = \int_G dx\ \mathrm{tr}\{F(x)H(x)^*\}.$$

Il résulte par ailleurs des travaux de plusieurs auteurs que, muni du produit de convolution

$$(F*H)(x) = \int_G dy\, F(y)H(xy^{-1}),$$

l'espace $C_c^{(\infty)}(G,K,\tau,\tau)$ devient une algèbre *commutative*, ce qui est une propriété essentielle pour en développer l'analyse.
Les faits suivants sont également de première importance : d'une part, en vertu de la décomposition de Cartan de G, une fonction τ-radiale est entièrement déterminée par sa restriction à A, et même à $\overline{A^+} \simeq \mathbb{R}_+$. D'autre part, il est facile de constater que

$$\forall a_t \in A, \quad F(a_t) \in \mathrm{End}_M\, \mathcal{H}_\tau.$$

Puisque $\tau_{|M}$ se décompose sans multiplicité, le lemme de Schur implique que $F(a_t)$ est scalaire sur chaque sous-espace M-invariant de $\mathcal{H}_\tau$. On peut ainsi écrire :

$$F(a_t) = \begin{pmatrix} f_{p-1}(t)\,\mathrm{Id}_{\sigma_{p-1}} & 0 \\ 0 & f_p(t)\,\mathrm{Id}_{\sigma_p} \end{pmatrix},$$

$$F(a_t) = \begin{pmatrix} f_{p-1}(t)\,\mathrm{Id}_{\sigma_{p-1}} & 0 & 0 \\ 0 & f_p^+(t)\,\mathrm{Id}_{\sigma_p^+} & 0 \\ 0 & 0 & f_p^-(t)\,\mathrm{Id}_{\sigma_p^-} \end{pmatrix},$$

$$F(a_t) = f(t)\ \mathrm{Id}_{\tau^\pm_{n/2}},$$

lorsque p est respectivement générique, égal à $\frac{n-1}{2}$ ou à $\frac{n}{2}$. Les fonctions (à valeurs complexes) $f_{p-1}, f_p, f_p^\pm, f$ introduites ci-dessus seront appelées *composantes scalaires* de F (nous reprenons en cela le vocabulaire de [BR89]) et comme nous l'avons signalé, elles déterminent complètement une fonction τ-radiale. De plus, lorsque p est générique ou $p = \frac{n}{2}$, les composantes scalaires sont paires ; si $p = \frac{n-1}{2}$, f_{p-1} est paire mais $f_p^\pm(-t) = f_p^\mp(t)$ (ces propriétés reflètent l'action du groupe de Weyl W sur A).

La prochaine notion essentielle que nous voulons introduire est celle de fonction τ-sphérique, qui nous permettra ensuite de définir et d'étudier la transformation de

Fourier sphérique. Comme dans le cas scalaire, les fonctions τ-sphériques seront des transformées de Poisson particulières.

Soit $\mathbb{D}(G, K, \tau)$ l'algèbre des opérateurs différentiels invariants (à gauche) agissant sur $C^\infty(G, K, \tau)$. Introduisons les notations classiques d, d^* et $\Delta = dd^* + d^*d$ pour désigner respectivement la différentielle extérieure, la codifférentielle extérieure et le Laplacien des p-formes sur la variété $H^n(\mathbb{R})$. Un résultat dû à Gaillard ([Gai92]) décrit explicitement les générateurs de $\mathbb{D}(G, K, \tau)$:

$$\mathbb{D}(G, K, \tau) = \begin{cases} \mathbb{C}[dd^*, d^*d] & \text{si } \tau = \tau_p \text{ avec } p \text{ générique,} \\ \mathbb{C}[dd^*, *d] & \text{si } \tau = \tau_{\frac{n-1}{2}}, \\ \mathbb{C}[\Delta_{| C^\infty(G,K,\tau^\pm_{n/2})}] & \text{si } \tau = \tau^\pm_{\frac{n}{2}}. \end{cases}$$

En particulier, $\mathbb{D}(G, K, \tau)$ est toujours *commutative* (mais ce fait découle aussi d'arguments plus généraux).
Nous dirons qu'une fonction τ-radiale Φ sur G est *τ-sphérique* si, pour tout $\xi \in \mathcal{H}_\tau$, $\Phi(\cdot)\xi$ est une fonction propre de $\mathbb{D}(G, K, \tau)$, i.e. s'il existe un caractère $\mathcal{X}$ de $\mathbb{D}(G, K, \tau)$ tel que

$$\forall D \in \mathbb{D}(G, K, \tau), \quad D\,\Phi(\cdot)\xi = \mathcal{X}(D)\,\Phi(\cdot)\xi.$$

Ouvrons une large parenthèse sur la transformation de Poisson des formes sur le bord K/M de G/K. Pour p générique, $\lambda \in \mathbb{C}$ et $\omega \in C^\infty(G, P, \sigma_p \otimes e^{i\lambda} \otimes 1)$ (i.e. ω est une p-forme sur $G/P = K/M$), soit

$$\phi_p^q(\lambda, x, \omega) = P_p^q \circ \pi_{\sigma_p,\lambda}(x^{-1})\omega \qquad \text{pour } q = p, p+1,$$

où P_p^q est la projection orthogonale (K-équivariante) de $\mathcal{H}_{\sigma_p,\lambda}$ sur $\mathcal{H}_{\tau_q}$, c'est-à-dire un générateur de $\mathrm{Hom}_K(\mathcal{H}_{\sigma_p,\lambda}, \mathcal{H}_{\tau_q})$, et est définie par

$$P_p^q(\omega) = \sqrt{\frac{\dim \tau_q}{\dim \sigma_p}} \int_K dk\, \tau_q(k)\, \omega(k).$$

Alors $\phi_p^q(\lambda, \cdot, \omega) \in C^\infty(G, K, \tau_q)$, et comme l'application

$$\begin{aligned} C^\infty(G, P, \sigma_p \otimes e^{i\lambda} \otimes 1) &\to C^\infty(G, K, \tau_q) \\ \omega &\mapsto \phi_p^q(\lambda, \cdot, \omega) \end{aligned}$$

est linéaire, continue et G-équivariante, $\phi_p^q(\lambda, \cdot, \omega)$ est la *transformée de Poisson* de ω. Lorsque $p = \frac{n-1}{2}$ ou $p = \frac{n}{2}$, en considérant $\sigma^\pm_{(n-1)/2}$ et $\tau^\pm_{n/2}$, on obtient de manière analogue des transformées de Poisson $\phi^q_{p,\pm}$ et $\phi_p^{q,\pm}$.

Des calculs (fastidieux) montrent qu'en fait, ces transformées de Poisson sont des fonctions propres pour les générateurs de $\mathbb{D}(G, K, \tau)$, et en particulier pour le Laplacien Δ (cf. Propositions 4.5 et 4.7.) : on a en effet

$$\Delta\phi(\lambda, \cdot, \omega) = \{\lambda^2 + (\rho - p)^2\}\phi(\lambda, \cdot, \omega)$$

pour chacune des fonctions $\phi = \phi_p^q, \phi_{p,\pm}^q, \phi_p^{q,\pm}$. [5]

Expliquons maintenant comment nous construisons les fonctions τ-sphériques sur G. Pour p générique, notons $J_p^q = (P_p^q)^*$ le plongement isométrique K-équivariant de $\mathcal{H}_{\tau_q}$ dans $\mathcal{H}_{\sigma_p,\lambda|_K} \simeq L^2(K, M, \sigma_p)$ défini par

$$J_p^q\xi = \sqrt{\frac{\dim \tau_q}{\dim \sigma_p}}\, P_{\sigma_p}\, \{\tau_q(\cdot)^{-1}\xi\},$$

où P_{σ_p} désigne la projection orthogonale de $\mathcal{H}_{\tau_q}$ sur $\mathcal{H}_{\sigma_p}$. De même, notons $J_{p,\pm}^q$ et $J_p^{q,\pm}$ les variantes dans les cas spéciaux.

Fixons dorénavant le degré p d'une forme sur G/K obtenue comme transformée de Poisson d'une forme de degré q sur le bord. D'après les résultats précédents, q ne peut prendre que les valeurs $p-1$ ou p. Nous mettons en garde le lecteur du fait que, par cette convention, les indices p et q seront échangés dans les expressions. Pour $\lambda \in \mathbb{C}$ et $x \in G$, posons

(i) si p est générique et $q = p - 1, p$:

$$\Phi_q^p(\lambda, x) = P_q^p \circ \pi_{\sigma_q,\lambda}(x^{-1}) \circ J_q^p;$$

(ii) si $p = \frac{n-1}{2}$:

$$\begin{cases} \Phi_{p-1}^p(\lambda, x) = P_{p-1}^p \circ \pi_{\sigma_{p-1},\lambda}(x^{-1}) \circ J_{p-1}^p, \\ \Phi_{p,\pm}^p(\lambda, x) = P_{p,\pm}^p \circ \pi_{\sigma_p^\pm,\lambda}(x^{-1}) \circ J_{p,\pm}^p; \end{cases}$$

(iii) si $p = \frac{n}{2}$:

$$\begin{aligned} \Phi^\pm(\lambda, x) &= P_{p-1}^{p,\pm} \circ \pi_{\sigma_{p-1},\lambda}(x^{-1}) \circ J_{p-1}^{p,\pm} \\ &= P_p^{p,\pm} \circ \pi_{\sigma_p,\lambda}(x^{-1}) \circ J_p^{p,\pm}. \end{aligned}$$

Puisque $\Phi_q^p(\lambda, x)\xi = \phi_q^p(\lambda, x, J_q^p\xi)$, on en déduit que $\Phi_q^p(\lambda, \cdot)$ est une fonction τ_p-sphérique sur G (de même pour les variantes). En réalité, les fonctions $\Phi(\lambda, \cdot)$ définies ci-dessus fournissent *toutes* les fonctions τ-sphériques sur G. Plus précisément, on a le résumé suivant (cf §5.2).

[5] A l'aide du Théorème 0.5, on en déduit aussitôt le spectre L^2 du Laplacien Δ.

Théorème 0.6. *Soit* Φ *une fonction* τ*-radiale normalisée (i.e.* $\Phi(e) = \mathrm{Id}$*), et soit* $\Sigma(G, K, \tau, \tau)$ *l'ensemble des fonctions* τ*-sphériques sur* G*.*

(I) *Soit* p *générique. Alors :*

(i) $\Phi = \Phi^p_{p-1}(\lambda, \cdot) \Longleftrightarrow \begin{cases} \Delta\Phi = \{\lambda^2 + [\rho - (p-1)]^2\}\Phi, \\ d\Phi = 0; \end{cases}$

(ii) $\Phi = \Phi^p_p(\lambda, \cdot) \Longleftrightarrow \begin{cases} \Delta\Phi = \{\lambda^2 + (\rho - p)^2\}\Phi, \\ d^*\Phi = 0; \end{cases}$

(iii) $\Sigma(G, K, \tau_p, \tau_p) = \{\Phi^p_q(\lambda, \cdot) : q = p-1, p \text{ et } \lambda \in \mathbb{C}/\pm 1\}$.

(II) *Soit* $p = \frac{n-1}{2}$*. Alors :*

(i) $\Phi = \Phi^p_{p-1}(\lambda, \cdot) \Longleftrightarrow \begin{cases} \Delta\Phi = (\lambda^2 + 1)\Phi, \\ d\Phi = 0; \end{cases}$

(ii) $\Phi = \Phi^p_{p,\pm}(\lambda, \cdot) \Longleftrightarrow \begin{cases} *d\Phi = \pm i^{p^2-1}\,\lambda\,\Phi, \\ dd^*\Phi = 0, \end{cases}$ $\qquad (\Longrightarrow \Delta\Phi = \lambda^2\,\Phi)$ *;*

(iii) $\Sigma(G, K, \tau_p, \tau_p) = \{\Phi^p_{p-1}(\lambda, \cdot), \Phi^p_{p,\pm}(\lambda, \cdot) : \lambda \in \mathbb{C}/\pm 1\}$.

(III) *Soit* $p = \frac{n}{2}$*. Alors :*

(i) $\Phi = \Phi^\pm(\lambda, \cdot) \Longleftrightarrow \Delta\Phi = (\lambda^2 + \frac{1}{4})\Phi$ *;*

(ii) $\Sigma(G, K, \tau^\pm_p, \tau^\pm_p) = \{\Phi^\pm(\lambda, \cdot) : \lambda \in \mathbb{C}/\pm 1\}$.

De plus, $\Phi \in L^2(G, K, \tau, \tau)_\Delta \Longleftrightarrow \tau = \tau^\pm_{\frac{n}{2}}$ *et* $\Phi = \Phi^\pm(\frac{i}{2}, \cdot) = \Phi^\pm(-\frac{i}{2}, \cdot)$.

L'argument principal intervenant dans la démonstration de ce théorème est le fait remarquable que, comme dans le cas scalaire, les fonctions τ-sphériques que nous avons définies — ou plus précisément, leurs composantes scalaires — s'expriment en termes de fonctions de Jacobi $\phi^{(\alpha,\beta)}_\lambda$. Prenons un exemple : lorsque p est générique, les composantes scalaires $\varphi_{p-1}(\lambda, \cdot)$ et $\varphi_p(\lambda, \cdot)$ de $\Phi^p_{p-1}(\lambda, \cdot)$ s'écrivent :

$$\begin{aligned} \varphi_{p-1}(\lambda, t) &= \tfrac{n}{p}\phi^{(\frac{n}{2}-1, -\frac{1}{2})}_\lambda(t) - \tfrac{n-p}{p}(\operatorname{ch} t)\phi^{(\frac{n}{2}, -\frac{1}{2})}_\lambda(t), \\ \varphi_p(\lambda, t) &= \phi^{(\frac{n}{2}, -\frac{1}{2})}_\lambda(t), \end{aligned}$$

tandis que celles de $\Phi^p_p(\lambda, \cdot)$ sont :

$$\begin{aligned} \varphi_{p-1}(\lambda, t) &= \phi^{(\frac{n}{2}, -\frac{1}{2})}_\lambda(t), \\ \varphi_p(\lambda, t) &= \tfrac{n}{n-p}\phi^{(\frac{n}{2}-1, -\frac{1}{2})}_\lambda(t) - \tfrac{p}{n-p}(\operatorname{ch} t)\phi^{(\frac{n}{2}, -\frac{1}{2})}_\lambda(t). \end{aligned}$$

Par ailleurs, nous obtenons également l'expression des fonctions τ-sphériques comme intégrales d'Eisenstein : (toujours dans le cas générique)

$$\Phi^p_q(\lambda, x) = \frac{\dim \tau_p}{\dim \sigma_q} \int_K dk\, e^{-(i\lambda+\rho)H(xk)} \tau_p(k) \circ P_{\sigma_q} \circ \tau_p(\underline{k}(xk))^{-1}$$

(expression qui généralise celle du cas scalaire (0.7)).

Nous voilà maintenant parés pour définir la transformation de Fourier (τ-)sphérique. Cette transformation de Fourier des fonctions τ-radiales est naturellement définie comme la transformation de Gelfand de l'algèbre commutative $C_c(G, K, \tau, \tau)$. Un résultat (général) de notre Appendice B (Lemme B.1) montrant que tout caractère de Gelfand de cette algèbre s'écrit

$$\mathcal{H}(F)(\Phi) = \frac{1}{\dim \tau} \int_G dx \; \mathrm{tr}\{F(x)\Phi(x^{-1})\}$$

pour $F \in C_c(G, K, \tau, \tau)$ et $\Phi \in \Sigma(G, K, \tau, \tau)$, nous sommes amenés à définir deux transformations de Fourier τ-sphériques dans le cas générique

$$\mathcal{H}_q^p(F)(\lambda) = \frac{1}{C_n^p} \int_G dx \; \mathrm{tr}\{F(x)\Phi_q^p(\lambda, x^{-1})\} \qquad (q = p-1, p),$$

et de la même façon trois transformations $\mathcal{H}_{p-1}^p, \mathcal{H}_{p,\pm}^p$ dans le cas $p = \frac{n-1}{2}$, et deux transformations $\mathcal{H}^{\pm}$ dans le cas $p = \frac{n}{2}$.
Par les remarques que nous avons faites plus haut, toutes ces transformations de Fourier sphériques se ramènent *grosso modo* à des transformations de Jacobi. En utilisant les résultats de Flensted-Jensen et Koornwinder, nous en déduisons les théorèmes de Plancherel visés. Par commodité, nous laissons de côté le cas $p = \frac{n-1}{2}$, où les expressions sont similaires au cas générique.

Théorème 0.7 (Théorème de Plancherel pour $L^2(G, K, \tau_p, \tau_p)$). *Avec les notations précédentes,*

(I) *Soit p générique. Alors il existe deux mesures de Plancherel $d\nu_{p-1}(\lambda)$ et $d\nu_p(\lambda)$ sur $\mathbb{R}$ telles que :*

(i) *la transformation de Fourier des fonctions $F \in C_c^\infty(G, K, \tau_p, \tau_p)$ s'inverse par la formule*

$$F(x) = \sum_{q=p-1,p} \int_0^\infty d\nu_{p-1}(\lambda)\, \mathcal{H}_q^p(F)(\lambda)\, \Phi_q^p(\lambda, x);$$

(ii) *la transformation de Fourier τ_p-sphérique s'étend en une isométrie bijective entre espaces L^2.*

(II) *Soit $p = \frac{n}{2}$. Alors il existe une mesure de Plancherel $d\nu(\lambda)$ sur $\mathbb{R}$ telle que :*

(i) *la transformation de Fourier des fonctions* $F^{\pm} \in C_c^\infty(G, K, \tau_p^{\pm}, \tau_p^{\pm})$ *s'inverse par la formule*

$$F^{\pm}(x) = \int_0^\infty d\nu(\lambda)\, \mathcal{H}^{\pm}(F^{\pm})(\lambda)\, \Phi^{\pm}(\lambda, x) + 2^{1-2n}\, n\, \mathcal{H}^{\pm}(F^{\pm})(\tfrac{i}{2})\, \Phi^{\pm}(\tfrac{i}{2}, x);$$

(ii) *la transformation de Fourier* $\tau_p^{\pm}$*-sphérique s'étend en une isométrie bijective entre espaces* L^2.

De plus, nous avons également obtenu la description de l'image de la transformation de Fourier τ-sphérique, aussi bien dans le cadre des fonctions C^∞ à support compact (théorème de Paley-Wiener) que dans celui des espaces de Schwartz. Les mesures de Plancherel sont calculées explicitement (et nous retrouvons ainsi les résultats de [vdV93] et [CH94]). Par exemple, dans le cas générique,

$$d\nu_{p-1}(\lambda) = \frac{2}{\pi} \frac{np}{\lambda^2 + [\rho - (p-1)]^2} \frac{d\lambda}{|c(\lambda)|^2},$$
$$d\nu_p(\lambda) = \frac{2}{\pi} \frac{n(n-p)}{\lambda^2 + (\rho - p)^2} \frac{d\lambda}{|c(\lambda)|^2},$$

où la fonction c de Harish-Chandra est donnée par

$$c(\lambda) = \frac{2^n \Gamma(\frac{n}{2}+1)}{\sqrt{\pi}} \frac{\Gamma(i\lambda)}{\Gamma(i\lambda + \frac{n+1}{2})}.$$

Définissons maintenant la transformation de Fourier d'une fonction de type τ sur G. Si $\pi \in \widehat{G}$ est une représentation intervenant dans la décomposition abstraite de $L^2(G, K, \tau)$ (cf. Théorème 0.5), nous posons

$$\mathcal{H}(f)(\pi) = \frac{1}{\dim \tau} \int_G dx\, \pi(x) \circ J_\pi^\tau f(x), \tag{0.14}$$

(où J_π^τ est un générateur bien choisi de $\mathrm{Hom}_K(\mathcal{H}_\tau, \mathcal{H}_\pi)$) qui généralise à l'évidence la définition (0.5) donnée dans le cas scalaire, lorsque π est une série principale : en effet, dans le cas générique nous obtenons en considérant $\pi = \pi_{\sigma_q, \lambda}$:

$$\begin{aligned}\mathcal{H}_q^p(f)(\lambda, k) &= \frac{1}{C_n^p} \int_G dx\, \{\pi_{\sigma_q,\lambda}(x) J_q^p f(x)\}(k),\\ &= \frac{1}{\sqrt{C_n^p\, C_{n-1}^q}} \int_G dx\, e^{-(i\lambda+\rho)H(x^{-1}k)} P_{\sigma_q}\, \tau_p(\underline{k}(x^{-1}k))^{-1} f(x),\end{aligned}$$

et $\mathcal{H}_q^p(f)(\lambda, \cdot)$ est un élément de $C^\infty(K, M, \sigma_q)$ dès que $f \in C_c^\infty(G, K, \tau_p)$. Bien entendu, les définitions sont similaires dans les cas spéciaux. L'analyse sphérique développée précédemment conduit alors au résultat suivant (nous laissons de nouveau de côté le cas $p = \frac{n-1}{2}$).

Théorème 0.8 (Théorème de Plancherel pour $L^2(G,K,\tau_p)$). *Avec les notations précédentes,*

(I) *Soit p générique. Alors :*

(i) *la transformation de Fourier des fonctions $f \in C_c^\infty(G,K,\tau_p)$ s'inverse par la formule*

$$f(x) = C_n^p \sum_{q=p-1,p} \int_0^\infty d\nu_q(\lambda)\, P_q^p\, \pi_{\sigma_q,\lambda}(x^{-1})\, \mathcal{H}_q^p(f)(\lambda,\cdot);$$

(ii) *la transformation de Fourier s'étend en une isométrie bijective entre espaces L^2.*

(II) *Soit $p = \frac{n}{2}$ et soit $q = p-1$ ou $q = p$. Alors :*

(i) *la transformation de Fourier des fonctions $f^\pm \in C_c^\infty(G,K,\tau_p^\pm)$ s'inverse par la formule*

$$f^\pm(x) = C_n^{n/2}\Big\{\int_0^\infty d\nu(\lambda)\, P^\pm\, \pi_{\sigma_q,\lambda}(x^{-1})\mathcal{H}(f)(\lambda,\cdot) + 2^{1-2n}\, n\, P^\pm\, \pi_{\sigma_q,-\frac{i}{2}}(x^{-1})\mathcal{H}^\pm(f^\pm)(\tfrac{i}{2},\cdot)\Big\};$$

(ii) *la transformation de Fourier s'étend en une isométrie bijective entre espaces L^2, et la formule de Plancherel s'écrit*

$$\|f^\pm\|_{L^2}^2 = (C_n^{n/2})^2\Big\{\int_0^\infty d\nu(\lambda)\, \|\mathcal{H}^\pm(f^\pm)(\lambda,\cdot)\|_{L^2(K,M,\sigma_q)}^2 + 2^{1-2n}\, n\, \|\mathcal{H}^\pm(f^\pm)(\pi^\pm)\|_{\mathcal{H}_{\pi^\pm}}^2\Big\},$$

où $\pi^\pm$ désignent les séries discrètes du Théorème 0.5.

En particulier, on retrouve le Théorème de Plancherel abstrait 0.5, puisque

$$L^2(G,K,\tau_{\frac{n}{2}}) = L^2(G,K,\tau_{\frac{n}{2}}^+) \oplus L^2(G,K,\tau_{\frac{n}{2}}^-).$$

Notons que, dans le cas $p = \frac{n}{2}$, le théorème de Plancherel n'est pas une conséquence immédiate de la formule d'inversion ; sa démonstration nécessite en particulier d'utiliser le *plongement* des séries discrètes $\pi^\pm$ dans la série principale *non unitaire* $\pi_{\sigma_q,-\frac{i}{2}}$ afin d'identifier certains produits scalaires.

Le théorème précédent complète la liste des résultats classiques que nous escomptions de l'analyse harmonique des formes différentielles sur l'espace hyperbolique réel (au théorème de Paley-Wiener près). Nous donnons dans les deux derniers paragraphes (7 et 8) des applications de la théorie sphérique, principalement :

— la généralisation de la *transformation d'Abel* au fibré des formes différentielles sur $H^n(\mathbb{R})$, avec notamment des expressions explicites pour cette transformation ainsi que pour son inverse ;
— l'expression explicite du *noyau de la chaleur* associé au Laplacien des formes différentielles sur $H^n(\mathbb{R})$ (sans toutefois donner d'estimation de son comportement) ;
— le calcul rapide d'invariants géométrico-topologiques appelés *invariants de Novikov-Shubin*, qui avaient d'abord été déterminés d'une autre façon dans [Lot92].

Nous avons ensuite consacré deux appendices à des résultats généraux, dont le cadre dépasse celui que nous avons fixé jusqu'à présent.

L'Appendice A reformule et précise un résultat de [Bor85] qui décrit quelles séries discrètes interviennent dans la décomposition du fibré L^2 des formes différentielles sur un espace symétrique riemannien de type non compact G/K général (i.e. de rang quelconque).

L'Appendice B est dédié à la théorie « abstraite » des fonctions sphériques sur ce que nous appelons un *triplet de Gelfand* (G, K, τ). Rappelons que, lorsque (G, K) est une paire de Gelfand, i.e. lorsque l'algèbre $C_c(G)^\natural$ est commutative pour le produit de convolution, il est bien connu qu'on peut développer une théorie abstraite des fonctions sphériques (jusqu'à la formule de Plancherel) en donnant notamment plusieurs caractérisations équivalentes des fonctions sphériques. Nous généralisons ces résultats au cadre suivant : G est un groupe localement compact unimodulaire, K un sous-groupe compact de G et τ une représentation unitaire irréductible quelconque de K tels que l'algèbre $C_c(G, K, \tau, \tau)$ soit commutative pour le produit de convolution — dans ce cas, nous disons que (G, K, τ) est un triplet de Gelfand. Deitmar ([Dei90]) a fourni une classe importante d'exemples de ce type parmi les espaces symétriques non compacts G/K, en montrant par exemple que la condition précédente est équivalente au fait que $\tau_{|M}$ se décompose en facteurs irréductibles sans multiplicité.
De plus, lorsque G est un groupe de Lie semi-simple, connexe, non compact et de centre fini, et K un sous-groupe compact maximal de G (ainsi G/K est un espace symétrique de type non compact général), nous donnons une caractérisation

supplémentaire des fonctions τ-sphériques, nous montrons que *toutes* les fonctions τ-sphériques sur G peuvent se définir à partir des séries principales de G, et nous relions leur comportement asymptotique à la fonction c (généralisée) de Harish-Chandra, retrouvant ainsi un phénomène connu dans le cas scalaire.

En guise de conclusion, nous mentionnons qu'il nous paraît raisonnable d'envisager des résultats similaires à ceux que nous avons développés dans cette thèse lorsqu'on considère les espaces hyperboliques complexe et quaternionique au lieu de l'espace hyperbolique réel. Nous venons seulement d'initier les recherches dans cette direction. L'obtention d'un théorème de Plancherel « abstrait », ainsi que la détermination du spectre du Laplacien sur les formes constituent nos premiers objectifs. La complexité apparente de certains calculs ne permet pas de nous avancer davantage en ce qui concerne l'approche analytique « concrète » de la théorie.

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Harmonic Analysis for Differential Forms on Real Hyperbolic Spaces

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Abstract. We present the L^2 harmonic analysis for differential forms on the real hyperbolic space $H^n(\mathbb{R})$. The classical notions and results for functions on $H^n(\mathbb{R})$ are generalized: Poisson transforms, spherical functions, spherical Fourier transform, Fourier transform. Some basic applications are derived: study of the Abel transform and exact expression for the heat kernel. Two appendices are dedicated to more general results: a description of discrete series of representations occuring in the L^2 decomposition of differential forms on general Riemannian symmetric spaces of noncompact type G/K and a matrix-valued generalization of the spherical function theory for Gelfand pairs.

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Contents

1 Introduction

Harmonic analysis on Riemannian symmetric spaces of noncompact type G/K constitutes a nice chapter of the contemporary representation theory of semisimple Lie groups. Since its first developments in the 1950's with Harish-Chandra, it has been studied intensively during the last decades — mainly by Helgason — and is now in some sense 'complete' (nowadays, the efforts seem rather led toward the pseudo-Riemannian case). Nice surveys are presented in [Hel84, Hel94], and, for the rank-one case, in [Far82], [Koo84] and [Ank93].

A natural extension of this theory should be its study on (homogeneous) vector bundles over G/K. Several directions have already been investigated in that context: [Shi90, Shi94] and [Cam97a, Cam97b] are the first attempts to generalize the whole harmonic analysis on G/K to bundles over it, but some independent parts of the theory have been examined more intensively, such as vector-valued Poisson transforms ([Gai86, Gai88], [Olb94], [Ven93, Ven94], [Yan94]), the spherical functions theory ([War72], [Rad76], [Min92]), or a vector-valued Radon transform applied to differential equations ([BOS94]).

One of the most interesting and natural examples is the bundle of differential forms on G/K, because of its obvious connections with cohomology theory, geometry, topology and theoretical physics (see e.g. [VZ84], [Bor85], [Lot92], [CH94]). In particular, when $G/K = SO_e(n,1)/SO(n)$ is the real hyperbolic space $H^n(\mathbb{R})$, notable progresses have been made, concerning again Poisson transforms and differential operators ([Don81], [Gai86], [Str89], [CH96], [BOS94], [Ale95]) and the calculation of the Plancherel measure ([Ven93] and [CH94]). But the first thorough study of Fourier analysis in this setting is due to Badertscher and Reimann [BR89], who considered the case of vector fields (i.e. 1-forms) on $H^n(\mathbb{R})$.

The purpose of this paper is precisely to generalize their work, namely to realize the Fourier decomposition of the space $L^2 \wedge^p H^n(\mathbb{R})$ of square-integrable differential forms of degree p on the n-dimensional real hyperbolic space — in other words: to obtain an explicit Plancherel theorem for $L^2 \wedge^p H^n(\mathbb{R})$. As a matter of fact, we globally follow the spirit of [BR89], although we adopt sometimes a more theoretical point of view. Unsurprisingly, our results have strong similarities with those concerning functions (0-forms) and vector fields (1-forms), the most impressive (and heavy with consequences) being maybe the expression of the spherical functions in terms of the so-called Jacobi functions. These analogies make easier the understanding of what happens for p-forms of arbitrary degree, even though the final presentation might look lengthy or delicate to handle — especially because we were led to treat separately at each step three classes of values of p, as we shall see soon.

We come now to the organization of our paper. From now on, we will suppose $G = SO_e(n,1)$, $K = SO(n)$ and $G/K = H^n(\mathbb{R})$, except where indicated.

In Section 2, we first recall the basic (and standard) notation: structure of G, normalization of measures, identification between $L^2 \wedge^p H^n(\mathbb{R})$ and the space $L^2(G,K,\tau_p)$ of square-integrable functions $f : G \to \wedge^p \mathbb{C}^n$ of *right type* τ_p, i.e. such that

$$f(gk) = \tau_p(k)^{-1} f(g) \qquad (\forall g \in G,\ \forall k \in K),$$

where $\tau_p = \wedge^p \operatorname{Ad}^*$ denotes the coadjoint representation of K on $\wedge^p(\mathfrak{g}_\mathbb{C}/\mathfrak{k}_\mathbb{C})^* \simeq \wedge^p \mathbb{C}^n$. Then, an obvious remark is that the Hodge duality allows us to restrict our study to the degrees $0 \leqslant p \leqslant \frac{n}{2}$, and we conclude the section by presenting two results concerning differential operators: Gaillard's beautiful description of the algebra $\mathbb{D}(G,K,\tau_p)$ of left-invariant operators acting on $C^\infty \wedge^p H^n(\mathbb{R}) \simeq C^\infty(G,K,\tau_p)$, and the Hodge-de Rham decomposition of $L^2(G,K,\tau_p)$.

In Section 3, we develop an 'abstract approach' to the Plancherel theorem for the space $L^2(G,K,\tau_p)$ based on Harish-Chandra's famous Plancherel theorem for $L^2(G)$. Precisely, we show that if $p < \frac{n-1}{2}$, resp. $p = \frac{n-1}{2}$, $L^2(G,K,\tau_p)$ decomposes as the direct integral of two, resp. three inequivalent principal series of G, while if $p = \frac{n}{2}$, $L^2(G,K,\tau_p)$ decomposes as the direct integral of two equivalent principal series *plus* two discrete series representations of G. This latter exceptional fact is a consequence of a more general result due to Borel and developed in Appendix A.

In Section 4, we introduce in a natural way Poisson transforms on $C^\infty \wedge^p H^n(\mathbb{R})$ and we prove that they are precisely eigenforms for the generators of $\mathbb{D}(G,K,\tau_p)$. This result is obtained by two complementary methods, one algebraic and one analytic, the first leading also to the complete description of the L^2 spectra of the Hodge-de Rham and Bochner Laplacians.

Section 5 is devoted to the study of what we call τ*-spherical functions*, when τ is one of the irreducible representations $\tau_1, \ldots, \tau_{\frac{n-1}{2}}, \tau^{\pm}_{\frac{n}{2}}$. These are τ*-radial functions*, i.e. matrix-valued functions $F : G \to \operatorname{End} \mathcal{H}_\tau$ such that

$$F(k_1 g k_2) = \tau(k_2)^{-1} F(g) \tau(k_1)^{-1} \qquad (\forall g \in G,\ \forall k_1, k_2 \in K),$$

which are furthermore eigenfunctions (in a certain sense) for the algebra $\mathbb{D}(G, K, \tau_p)$ — they can be also defined differently, see Appendix B. We first observe that τ-radial functions F are completely determined (according to the nature of τ) by one, two or three scalar functions — that we call, following [BR89], *scalar components of* F. Then, we produce τ-spherical functions on G by using the Poisson transforms we considered in Section 4, and a central result of our paper is that the scalar components of τ-spherical functions are exactly (linear combinations of) 'generalized' Jacobi functions (i.e. particular hypergeometric functions). This result was strongly sought for, since we knew that spherical functions are exactly Jacobi functions when $p = 0$. As a corollary, the set of τ-spherical functions on G is completely determined.

Another fortunate consequence is that the τ-spherical Fourier analysis is closely related to Jacobi analysis. At this point, we can then use the general theory developed by Flensted-Jensen and Koornwinder (see [Koo84]) to obtain an inversion formula and the Plancherel theorem for the spherical transform, as well as a Paley-Wiener theorem and an isomorphism between Schwartz spaces. This makes up the first part of Section 6. Then, using classical arguments, we show that the knowledge of the spherical transform suffices to derive the inversion formula and the Plancherel theorem for the Fourier transform of differential forms on $H^n(\mathbb{R})$. These results were, in a way, the main goal of our work, and they almost settle, up to a conjectural Paley-Wiener theorem, the basic Fourier analysis on the space $L^2(G, K, \tau_p)$.

In the last two sections, we present some interesting applications. In Section 7, we obtain explicit expressions for the Abel transform of τ-radial functions and for its inverse as combinations of differential and integral operators, generalizing again the well-known case $p = 0$. We also prove that the spherical transform is still the composition of the Abel transform followed by the Euclidean Fourier transform. This fact is used later in Section 8 to get explicit expressions for the heat kernel on the bundle $\wedge^p H^n(\mathbb{R})$ [1]. The last immediate application we give is the calculation of the *Novikov-Shubin invariants* of closed oriented real hyperbolic manifolds. This geometrical result had been first established in [Lot92].

Finally, we have added two appendices to our work. In Appendix A, we determine the discrete series contribution to $L^2 \wedge^p (G/K)$, for any Riemannian symmetric space

[1] In the case of bundles over odd dimensional real hyperbolic spaces, other classical P.D.E.'s (Maxwell, Dirac and spinor wave equations) have been studied in [BOS94].

of noncompact type G/K — this characterization appeared first in [Bor85], but we present here an alternative proof which was explained to us by Jean-Philippe Anker, and which looks simpler in our opinion. We give also a related result, concerning the eigenvalue description of the Laplacian on $L^2 \wedge^p (G/K)$.

In Appendix B, we develop an 'abstract theory' for τ-spherical functions on what we call a *Gelfand triple* (G, K, τ). Let G be a unimodular locally compact group, K a compact subgroup of G and τ an irreducible unitary representation of K. Then we say that (G, K, τ) is a Gelfand triple if the convolution algebra $C_c(G, K, \tau, \tau)$ of continuous compactly supported τ-radial functions on G is commutative. This object is intended then to be a natural generalization of the notion of Gelfand pairs, and includes as an example certain homogeneous vector bundles $G \times_K \mathcal{H}_\tau$ over a general noncompact Riemannian symmetric space G/K (and, in particular our bundle $\wedge^p H^n(\mathbb{R})$). The main result consists in establishing in this setting the analogues of the classical characterizations of spherical functions on Gelfand pairs (see e.g. [Far82] or [Ank93]). As a consequence, the spherical Fourier transform of τ-radial functions can be interpreted as the Gelfand transform of the commutative algebra $C_c(G, K, \tau, \tau)$. Let us mention that a similar theory has been carried out simultaneously in [Cam97b]. In addition, we describe explicitely the set of τ-spherical functions on a semisimple, connected, noncompact Lie group G with finite centre, and we relate their asymptotic behaviour to matrix-valued Harish-Chandra's c functions.

The work I present here constitutes my Ph.D. thesis, which was supervised by J.-Ph. Anker. I would like to express him my sincere gratitude for his guidance and constant encouragement, and it is obvious that his contribution to this work was essential. I am grateful to Roberto Camporesi for having read carefully parts of this manuscript and for his help in the proof of Proposition 6.26. I would like also to thank Pierre-Yves Gaillard, Toshiyuki Kobayashi and Robert Stanton for interesting conversations, as well as Mogens Flensted-Jensen and all the staff at the Mittag-Leffler Institute, where a substantial part of this work was done during the month of September 1995 thanks to the perfect organization.

Lastly, I mention that I am currently working on the adaptation of this paper — as far as possible — to the cases of complex and quaternionic hyperbolic spaces.

2 Differential forms on real hyperbolic spaces

2.1 Structure of real hyperbolic spaces

Generalities

On $(\mathbb{R}^{n+1})^2 = (\mathbb{R}^n \times \mathbb{R})^2$, define the Lorentz form L by:

$$L(x, y) = x_1 y_1 + \cdots + x_n y_n - x_{n+1} y_{n+1}.$$

The real hyperbolic space $H^n(\mathbb{R})$ is, by definition, the hyperboloid $\{x \in \mathbb{R}^{n+1} : L(x, x) = -1, x_{n+1} > 0\}$ equipped with the metric induced by $ds^2 = dx_1^2 + \cdots + dx_n^2 - dx_{n+1}^2$, so that it is a Riemannian manifold with constant sectional curvature -1. As usual, we shall view $H^n(\mathbb{R})$ as the symmetric space of noncompact type and of rank one G/K, where $G = SO_e(n, 1)$ and $K \simeq SO(n) \subset G$ is the stabilizer of the point $(0, \dots, 0, 1) \in \mathbb{R}^{n+1}$.

The Lie algebra $\mathfrak{g}$ of G decomposes as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k} = \mathfrak{so}(n)$ is the Lie algebra of K, and $\mathfrak{p} \simeq \mathbb{R}^n$ its orthogonal complement with respect to the Killing form $B(\cdot\,,\cdot)$. In $\mathfrak{p} = \left\{ \begin{pmatrix} 0 & y \\ {}^t y & 0 \end{pmatrix},\ y \in \mathbb{R}^n \right\}$, consider the element:

$$C_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}_{\underset{1}{\leftrightarrow}\ \underset{n-1}{\longleftrightarrow}\ \underset{1}{\leftrightarrow}}. \tag{2.1}$$

$\mathfrak{a} = \mathbb{R}\, C_0$ is a maximal abelian subspace of $\mathfrak{p}$. If

$$a_t = \exp(tC_0) = \begin{pmatrix} \operatorname{ch} t & 0 & \operatorname{sh} t \\ 0 & I_{n-1} & 0 \\ \operatorname{sh} t & 0 & \operatorname{ch} t \end{pmatrix},$$

then $A = \{a_t,\ t \in \mathbb{R}\}$ is a (closed) Lie subgroup of G, with Lie algebra $\mathfrak{a}$. Any element $g \in G$ can be written $g = k_1 a_t k_2$, with $k_1, k_2 \in K$, and a unique $t \geqslant 0$ (this is the Cartan decomposition of G).

Let $\operatorname{ad} : \mathfrak{g} \to \operatorname{End}(\mathfrak{g})$ be the adjoint representation of $\mathfrak{g}$. $\operatorname{ad} C_0$ is a diagonalizable endomorphism whose eigenvalues are 0 and ± 1; if $\mathfrak{n}$ denotes the eigenspace corresponding to the eigenvalue $+1$, then:

(i) $\mathfrak{n} = \left\{ \begin{pmatrix} 0 & {}^t y & 0 \\ -y & 0 & y \\ 0 & {}^t y & 0 \end{pmatrix},\ y \in \mathbb{R}^{n-1} \right\} \simeq \mathbb{R}^{n-1}$;

(ii) $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ (Iwasawa decomposition of $\mathfrak{g}$);

(iii) exp is a diffeomorphism from $\mathfrak{n}$ onto its image N, which is an abelian subgroup of G, normalized by A.

(iv) G is diffeomorphic to KAN, ANK or NAK (Iwasawa decompositions of G). When using the decomposition $G \simeq KAN$, we shall denote by H the Iwasawa projection on A such that $H(ka_t n) = t$, and by $\underline{k}$, resp. $\underline{n}$ the projections on K, resp. N.

Normalizations

We choose the different Haar measures on the groups in the following way:

- on K: we normalize dk so that $\int_K dk = 1$ (K is compact).
- on A: since $A = \{a_t,\ t \in \mathbb{R}\}$, we take dt to be the Lebesgue measure on $\mathbb{R}$.
- on $M := Z_K(A)$: $M \simeq SO(n-1)$ is compact, then dm is normalized so that $\int_M dm = 1$.
- on N: identifying N and $\mathfrak{n}$ via the exponential map, and using the identification given in (i) above, the measure dn on N is the Lebesgue measure on $\mathbb{R}^{n-1}$ [1].
- on G: according to the Iwasawa decomposition we may choose,

$$dg = e^{2\rho t} dk\, dt\, dn = dt\, dn\, dk = e^{-2\rho t} dn\, dt\, dk,$$

  where $\rho = \frac{n-1}{2}$ is the half-sum of the positive roots of $(\mathfrak{g}, \mathfrak{a})$.

The measure on $H^n(\mathbb{R})$ is given by $dx = dt\, dn$, since $G/K \simeq AN \cdot o$, where $o = eK$ is the origin of the hyperbolic space. Geometrically, this measure corresponds to the product $d\sigma \cdot (\operatorname{sh} t)^{n-1} dt$ of the respective measures on the unit sphere $\mathbb{S}^{n-1}$ and on $\mathbb{R}_+$ when one writes the decomposition in geodesical polar coordinates $G/K \simeq \mathfrak{p} \simeq K/M \times \mathfrak{a}^+$.

REMARK: In Section 5, we will change the normalization of the measure on $H^n(\mathbb{R})$.

[1] Note that this choice of measure is not the one generally adopted in the literature since Harish-Chandra's works. Here, the measure is defined so that it matches the normalization of the sectional curvature on G/K we have made previously.

2.2 The space $L^2(G, K, \tau_p)$

Let $0 \leqslant p \leqslant n$. A *differential p-form* on $H^n(\mathbb{R})$ is a section of the vector bundle $\wedge^p E$, where $E = T^*_{\mathbb{C}}(H^n(\mathbb{R}))$ is the complexified cotangent bundle of $H^n(\mathbb{R})$. We shall write

$$\Gamma \wedge^p H^n(\mathbb{R}) = \{\text{differential } p\text{-forms on } H^n(\mathbb{R})\},$$
$$\Gamma \wedge H^n(\mathbb{R}) = \{\text{differential forms on } H^n(\mathbb{R}) \text{ of any degree}\},$$

and shall replace 'Γ' by 'C', 'C_c', 'C^∞', 'C_c^∞', 'L^2', if the sections considered are respectively continuous, continuous with compact support, smooth, smooth with compact support, square-integrable. The Riemannian metric on $H^n(\mathbb{R})$ given previously induces a (local) Hermitian scalar product $(\cdot\,,\cdot)_x$ on the fibre $\wedge^p E_x$ of $\wedge^p E$ at the point $x \in H^n(\mathbb{R})$. Then, endowed with the (global) Hermitian scalar product

$$(\omega_1, \omega_2) = \int_{H^n(\mathbb{R})} dx\, (\omega_1(x), \omega_2(x))_x,$$

$L^2 \wedge^p H^n(\mathbb{R})$ is a Hilbert space.

If τ is a unitary finite dimensional representation of K on a Hilbert space $\mathcal{H}_\tau$, we say that a function $f : G \to \mathcal{H}_\tau$ is *of (right) type* τ if it verifies the relation

$$f(gk) = \tau(k)^{-1} f(g) \qquad (\forall g \in G,\ \forall k \in K). \tag{2.2}$$

(In other words, f is a section of the homogeneous bundle $G \times_K \mathcal{H}_\tau$ over G/K.) We shall denote by $\Gamma(G, K, \tau)$ the space of functions of type τ on G and, as above, shall change 'Γ' for 'C', 'C_c', 'C^∞', 'C_c^∞', 'L^2' when needed.

Let τ_p be the standard representation of K on the p-th exterior algebra $\wedge^p \mathbb{C}^n$ of $\mathbb{C}^n$. Then τ_p is equivalent to the p-th exterior product $\wedge^p \operatorname{Ad}^*$ of the coadjoint representation Ad^* of K on $\mathfrak{p}^*_{\mathbb{C}}$, and this allows us to identify by a well-known procedure (see for example [Wal73]) a p-form on $H^n(\mathbb{R})$ with a function of type τ_p on G. Thereafter, we shall always use the identification

$$\Gamma \wedge^p H^n(\mathbb{R}) \equiv \Gamma(G, K, \tau_p).$$

Note that the scalar product on $L^2(G, K, \tau)$ is given by

$$(f_1, f_2) = \int_G dg\, \big(f_1(g), f_2(g)\big)_{\mathcal{H}_\tau}.$$

2.3 Invariant differential operators

We denote by $\mathbb{D}(G, K, \tau)$ the algebra of (left-)invariant differential operators acting on $C^\infty(G, K, \tau)$.

The algebra $\mathbb{D}(G, K, \tau_p)$

We recall now a result, due to Gaillard, which describes the algebra $\mathbb{D}(G, K, \wedge \mathrm{Ad}^*)$ of (left-)invariant differential operators acting on C^∞ differential forms (of any degree $0 \leqslant p \leqslant n$) on the real hyperbolic space $H^n(\mathbb{R})$.

Using standard notations, we denote respectively by d, $*$ and d^* the (exterior) differential map, the Hodge operator and the (exterior) codifferential map on differential forms. Recall the following essential facts (see e.g. [War83]):

(i) $d : C^\infty \wedge^p H^n(\mathbb{R}) \to C^\infty \wedge^{p+1} H^n(\mathbb{R})$;

(ii) $* : \Gamma \wedge^p H^n(\mathbb{R}) \to \Gamma \wedge^{n-p} H^n(\mathbb{R})$ and $** = (-1)^{p(n-p)}\,\mathrm{Id}$; the Hodge duality will permit us to restrict our study to p-forms for $0 \leqslant p \leqslant n/2$;

(iii) $d^* : C^\infty \wedge^p H^n(\mathbb{R}) \to C^\infty \wedge^{p-1} H^n(\mathbb{R})$, $d^* = (-1)^{n(p+1)+1} {*d*}$ on C^∞ p-forms and d^* is the L^2 adjoint of d for the Riemannian metric on $H^n(\mathbb{R})$;

(iv) $\Delta = dd^* + d^*d$ is the *Hodge-de Rham Laplacian* on $C^\infty \wedge H^n(\mathbb{R})$.

Let $p_0, \ldots, p_n$ be the projections from $\Gamma \wedge H^n(\mathbb{R})$ onto its homogeneous components of degree $0, \ldots, n$.

Theorem 2.1 ([Gai92], Theorem 3.1). $\mathbb{D}(G, K, \wedge \mathrm{Ad}^*)$ *is the associative algebra with unit element generated by* $p_0, \ldots, p_n$, $*$ *and* d.

Corollary 2.2. *Let* $0 \leqslant p \leqslant n/2$. *Then:*

(i) *when* $p < \frac{n-1}{2}$, $\mathbb{D}(G, K, \tau_p)$ *is generated by* dd^* *and* d^*d;

(ii) *when* $p = \frac{n-1}{2}$, $\mathbb{D}(G, K, \tau_p)$ *is generated by* $*d$ *and* dd^*;

(iii) *when* $p = \frac{n}{2}$, $\mathbb{D}(G, K, \tau_p)$ *is generated by* $*$ *and* $d{*}d$, *and* $\mathbb{D}(G, K, \tau_p^\pm)$ *is generated by the sole Laplacian* Δ *restricted to* $C^\infty(G, K, \tau_p^\pm)$.

In particular, $\mathbb{D}(G, K, \tau_p)$ *is commutative except for* $p = \frac{n}{2}$ *(but, in this case,* $\mathbb{D}(G, K, \tau_{n/2}^\pm)$ *is commutative).*

(For convenience, the definition of $\tau_{n/2}^\pm$ will be given in Section 3.)

The Hodge-Kodaira decomposition of $L^2(G, K, \tau_p)$

We recall in our context a famous result of global analysis on Riemannian manifolds (see [Rha73]). Let $L^2(G, K, \tau_p)_\Delta$ be the space of p-forms ω on $H^n(\mathbb{R})$ that are L^2 and harmonic, i.e. such that $\Delta\omega = 0$.

Theorem 2.3. *For all integers $0 \leqslant p \leqslant n$, the Hilbert space $L^2(G, K, \tau_p)$ of square-integrable p-forms on $H^n(\mathbb{R})$ decomposes into the following orthogonal direct sum:*

$$L^2(G, K, \tau_p) = \overline{d(C_c^\infty(G, K, \tau_{p-1}))} \overset{\perp}{\oplus} \overline{d^*(C_c^\infty(G, K, \tau_{p+1}))} \overset{\perp}{\oplus} L^2(G, K, \tau_p)_\Delta.$$

Notice that $L^2(G, K, \tau_p)_\Delta = \{0\}$, except for n even and $p = \frac{n}{2}$. In the latter case, this space corresponds exactly to the contribution of the discrete series representations which decompose the space $L^2(G, K, \tau_p)$. We examine this point in detail in next section.

3 The abstract Plancherel Theorem for $L^2(G, K, \tau_p)$

3.1 Presentation

As a semisimple connected real Lie group with finite centre, $G = SO_e(n,1)$ is a unimodular 'type I' group, and the abstract Plancherel theorem for $L^2(G)$ (due independently to Mautner and Segal) gives the decomposition of this space into irreducible components under the action of the biregular representation of G: if $\widehat{G}$ denotes the unitary dual of G,

$$L^2(G) \simeq \int_{\widehat{G}}^{\oplus} d\nu(\pi)\, \mathcal{H}_\pi \,\widehat{\otimes}\, \mathcal{H}_\pi^*, \tag{3.1}$$

where $\mathcal{H}_\pi$ stands for the Hilbert space of the representation $\pi \in \widehat{G}$, $\mathcal{H}_\pi^*$ for its dual space and $d\nu$ for the Plancherel measure ($\widehat{\otimes}$ means Hilbert completion) [1]. The equivalence in (3.1) is realized by the Fourier transform:

$$f \mapsto \widehat{f}(\pi) = \int_G dx\, \pi(x) f(x).$$

The masterpiece of Harish-Chandra, culminating with [HC75, HC76a, HC76b], consisted in making (3.1) explicit in the context of Lie groups that are 'essentially' semisimple (see the definition of the Harish-Chandra class in [HC75]), giving notably a description of representations π occuring effectively in (3.1) and a formula for the Plancherel measure $d\nu$ (cf. for instance [HC75, HC76a, HC76b] or [Wal92] for precise statements).

In this section, our goal is to obtain in the same manner the decomposition of the space $L^2 \wedge^p H^n(\mathbb{R}) \simeq L^2(G, K, \tau_p)$ into irreducible factors.

Recall first that a linear map $T : \mathcal{H}_{\pi_1} \to \mathcal{H}_{\pi_2}$ between the spaces of two representations π_1, π_2 of a group G, such that

$$T \circ \pi_1(x) = \pi_2(x) \circ T \qquad (\forall\, x \in G) \tag{3.2}$$

is called an intertwining operator. The space of intertwining operators from $\mathcal{H}_{\pi_1}$ to $\mathcal{H}_{\pi_2}$ is denoted by $\mathrm{Hom}_G(\mathcal{H}_{\pi_1}, \mathcal{H}_{\pi_2})$, the subscript G referring then to the group on which the equivariant property (3.2) holds.

[1] As a reference for this result in the general setting, we mention [Wal92], Chapter 14.

As it has been remarked by several authors (see e.g. [AS77], [Bor85], [BOS94]), (3.1) provides an analogous decomposition of any module $L^2(G, K, \tau)$ induced from a finite dimensional unitary representation τ of K:

$$\begin{aligned} L^2(G, K, \tau) &= \{L^2(G) \otimes \mathcal{H}_\tau\}^K \\ &\simeq \int_{\widehat{G}}^{\oplus} d\nu(\pi)\, \mathcal{H}_\pi \,\widehat{\otimes}\, \{\mathcal{H}_\pi^* \otimes \mathcal{H}_\tau\}^K \\ &\simeq \int_{\widehat{G}}^{\oplus} d\nu(\pi)\, \mathcal{H}_\pi \,\widehat{\otimes}\, \mathrm{Hom}_K(\mathcal{H}_\pi, \mathcal{H}_\tau), \end{aligned} \tag{3.3}$$

where the upper index K stands for the space of K-invariant vectors, K acting here on the right on $L^2(G)$. We point out the fact that $\mathrm{Hom}_K(\mathcal{H}_\pi, \mathcal{H}_\tau)$ is necessarily finite dimensional, since every irreducible unitary representation of G is admissible.

As mentioned before, the support of the Plancherel measure $d\nu$ in (3.3) is not all of the set $\widehat{G}$; when G is of real rank one, only two particular types of representations of G do occur: the so-called principal series and discrete series. We recall briefly the definition of these two classes of representations.

For more complete results about (generalized) principal series representations, we refer to [Kna86], Chapter VII. We recall here their construction when G is of real rank one.

Let M be the centralizer of A in K (when $G = SO_e(n, 1)$, $M \simeq SO(n-1)$), and let $P = MAN$ be the usual (minimal) parabolic subgroup of G (associated with A and N). Given an irreducible unitary (then finite dimensional) representation σ of M on $\mathcal{H}_\sigma$ and $\lambda \in \mathbb{C}$, one can construct a representation $\sigma \otimes e^{i\lambda} \otimes 1$ of P on $\mathcal{H}_\sigma$ by putting:

$$(\sigma \otimes e^{i\lambda} \otimes 1)(m a_t n) = e^{i\lambda t} \sigma(m).$$

The *principal series representations* $\pi_{\sigma,\lambda}$ are, by definition, the representations induced from P to G:

$$\pi_{\sigma,\lambda} = \mathrm{Ind}_P^G(\sigma \otimes e^{i\lambda} \otimes 1).$$

Concretely, $\pi_{\sigma,\lambda}$ acts on the space

$$\begin{aligned} \mathcal{H}_{\sigma,\lambda} &= L^2(G, MAN, \sigma \otimes e^{i\lambda} \otimes 1) \\ &:= \{f : G \to \mathcal{H}_\sigma : f(g m a_t n) = e^{-(i\lambda+\rho)t} \sigma(m)^{-1} f(g),\ f_{|K} \in L^2(K)\} \end{aligned}$$

by left translations: $\pi_{\sigma,\lambda}(g)f(h) = f(g^{-1}h)$ — notice the ρ-shift —, where ρ is the half-sum of the positive roots of the pair $(\mathfrak{g}, \mathfrak{a})$ and equals $\frac{n-1}{2}$ when $G = SO_e(n,1)$. The restriction map $f \mapsto f_{|K}$ induces a bijective isometry from $\mathcal{H}_{\sigma,\lambda}$ onto the space $L^2(K, M, \sigma)$ of L^2 functions on K of (right) type σ, on which $\pi_{\sigma,\lambda}$ acts by

$$\pi_{\sigma,\lambda}(g)f(k) = e^{-(i\lambda+\rho)H(g^{-1}k)}\, f(\underline{k}(g^{-1}k)) \qquad (g \in G, k \in K).$$

Notice that $\pi_{\sigma,\lambda}$ is admissible (and unitary if λ is real); in particular, $\pi_{\sigma,\lambda|K} = \mathrm{Ind}_M^K(\sigma)$ — which acts precisely on $L^2(K, M, \sigma)$ by left translations — is always unitary. Moreover, the unitary principal series representations are irreducible (except maybe for $\lambda = 0$: see [HC76b], Lemma 13.3), and elements of the Weyl group $W(\mathfrak{g}, \mathfrak{a})$ can produce intertwining operators between such representations (see Bruhat's theorem in [Kna86], Theorem 7.2).

We recall now the definition of discrete series representations (see e.g. [Kna86]).

If π is a unitary representation of G on the space $\mathcal{H}_\pi$ endowed with a Hermitian scalar product $(\cdot\,,\cdot)$, a *matrix coefficient* of π is a function on G of the form $(\pi(\cdot)u, v)$, where $u, v \in \mathcal{H}_\pi$. When π is moreover irreducible, the following three conditions are equivalent:

(i) there exists a (nonzero) matrix coefficient of π which belongs to $L^2(G)$;

(ii) all matrix coefficients of π belong to $L^2(G)$;

(iii) π can be embedded in the (left or right) regular representation of G on $L^2(G)$.

If one of these conditions is satisfied, there exists a positive real number d_π such that

$$\int_G dx\, (\pi(x)u_1, v_1)\overline{(\pi(x)u_2, v_2)} = d_\pi^{-1}(u_1, u_2)\overline{(v_1, v_2)} \qquad (u_1, u_2, v_1, v_2 \in \mathcal{H}_\pi).$$

In this case, π is called a *discrete series representation of* G, and d_π is its *formal degree.*

It is well-known ([HC66], §39) that if G is semisimple, connected, with finite centre and contains a maximal compact subgroup K such that $\operatorname{rank} G = \operatorname{rank} K$, then G has discrete series representations; this fact corresponds to the existence of a compact Cartan subgroup of G (or, equivalently, of a compact Cartan subalgebra of $\mathfrak{g}$), which is a necessary and sufficient condition for the existence of discrete series [2].

[2] Recall that here, the (complex) rank of a Lie group is the common dimension of the Cartan subalgebras of its Lie algebra.

REMARK: in our context, the unique case where $G = SO_e(n,1)$ has discrete series representations occurs when n is even. Indeed, one has in this case $\operatorname{rank} G = \operatorname{rank} K = n/2$, while $\operatorname{rank} G = (n+1)/2$ and $\operatorname{rank} K = (n-1)/2$ when n is odd.

We are now able to make the decomposition (3.3) a little bit more explicit. In the case of a real rank one Lie group, Harish-Chandra's Plancherel Theorem (see for example [HC76b], [Bor85] or [Wal92], Theorem 13.4.1) allows us to develop (3.3) as:

$$L^2(G,K,\tau_p) \simeq \int_{W\backslash(\widehat{M}\times\mathfrak{a}^*)}^{\oplus} d\nu(\sigma,\lambda)\, \mathcal{H}_{\sigma,\lambda} \,\widehat{\otimes}\, \operatorname{Hom}_K(\mathcal{H}_{\sigma,\lambda}, \mathcal{H}_{\tau_p}) \\ \oplus \sum\nolimits_{\widehat{G}_d}^{\oplus} d_\pi\, \mathcal{H}_\pi \,\widehat{\otimes}\, \operatorname{Hom}_K(\mathcal{H}_\pi, \mathcal{H}_{\tau_p}), \tag{3.4}$$

where W is the Weyl group $W(\mathfrak{g},\mathfrak{a})$ and $\widehat{G}_d$ denotes the discrete part of $\widehat{G}$ [3]. In the sequel, we shall denote by $L^2(G,K,\tau_p)_c$ (respectively by $L^2(G,K,\tau_p)_d$) the continuous (respectively discrete) part of $L^2(G,K,\tau_p)$. In order to reduce as much as possible the decomposition (3.4), we must now determine whenever the spaces $\operatorname{Hom}_K(\mathcal{H}_{\sigma,\lambda}, \mathcal{H}_{\tau_p})$ and $\operatorname{Hom}_K(\mathcal{H}_\pi, \mathcal{H}_{\tau_p})$ are nontrivial.

3.2 Principal series representations decomposing $L^2(G,K,\tau_p)$

Recall that, since the principal series $\pi_{\sigma,\lambda}$ are induced representations from MAN to G, $\mathcal{H}_{\sigma,\lambda} \simeq L^2(K,M,\sigma)$ as a K-module. Thus, the variable λ disappears and $\operatorname{Hom}_K(\mathcal{H}_{\sigma,\lambda}, \mathcal{H}_{\tau_p}) \simeq \operatorname{Hom}_K(\mathcal{H}_{\tau_p}, L^2(K,M,\sigma)) \simeq \operatorname{Hom}_M(\mathcal{H}_\sigma, \mathcal{H}_{\tau_p})$, the last equivalence coming from the Frobenius reciprocity law: the multiplicity of τ_p in $\pi_{\sigma,\lambda|_K}$ is equal to the multiplicity of σ in $\tau_{p|_M}$. This explains why we need now to decompose τ_p and $\tau_{p|_M}$ into irreducible components. For the following facts, we refer, for instance, to [BD85] or [IT78].

K-decomposition of τ_p

(i) τ_p is irreducible if $p \neq n/2$.

(ii) $\tau_{\frac{n}{2}} = \tau_{\frac{n}{2}}^+ \oplus \tau_{\frac{n}{2}}^-$, the two factors being irreducible and inequivalent; they correspond to the decomposition $\wedge^{\frac{n}{2}}\mathbb{C}^n = \wedge_+^{\frac{n}{2}}\mathbb{C}^n \oplus \wedge_-^{\frac{n}{2}}\mathbb{C}^n$ of $\mathcal{H}_{\tau_{\frac{n}{2}}}$ into eigenspaces for the Hodge operator $*$. Precisely,

$$* = \pm i^{(\frac{n}{2})^2}\,\mathrm{Id} = \begin{cases} \pm\,\mathrm{Id} & \text{for } \frac{n}{2} \text{ even} \\ \pm i\,\mathrm{Id} & \text{for } \frac{n}{2} \text{ odd} \end{cases} \quad \text{on } \wedge_\pm^{\frac{n}{2}}\mathbb{C}^n. \tag{3.5}$$

[3] In (3.4), d_π is just a weight in $L^2(\widehat{G}_d)$, the multiplicity being implicit in $\operatorname{Hom}_K(\mathcal{H}_\pi, \mathcal{H}_{\tau_p})$.

(iii) The Hodge operator $*$ induces an equivalence $\tau_p \sim \tau_{n-p}$. We can therefore restrict to $0 \leqslant p \leqslant n/2$.

Determination of the dominant weights

Choose a Cartan subalgebra $\mathfrak{t}_{\mathbb{C}}$ of $\mathfrak{k}_{\mathbb{C}}$ (when n is odd, one can take for $\mathfrak{t}_{\mathbb{C}}$ a Cartan subalgebra of $\mathfrak{m}_{\mathbb{C}}$) and denote by $(e_i)_{i=1}^n$ the canonical basis of $\mathfrak{p} \simeq \mathbb{R}^n$.

- *case* $n = 2m + 1$ *odd:*

(i) elements of $\mathfrak{t}_{\mathbb{C}}$ [4]: $T_s = \begin{pmatrix} 0 & & & \\ & \begin{pmatrix} 0 & is_m \\ -is_m & 0 \end{pmatrix} & & \\ & & \ddots & \\ & & & \begin{pmatrix} 0 & is_1 \\ -is_1 & 0 \end{pmatrix} \\ & & & & 0 \end{pmatrix}$,

where $s = (s_1, \ldots, s_m) \in \mathbb{C}^m$;

(ii) positive roots: $\alpha_j \pm \alpha_k$ $(j < k)$, α_l, where $\alpha_j \in \mathfrak{t}_{\mathbb{C}}^*$ is defined by $\alpha_j(T_s) = s_j$;

(iii) positive Weyl chamber: $\{T_s \in \mathfrak{t}_{\mathbb{C}}^* : s_j \in \mathbb{R},\ s_1 > s_2 > \ldots > s_m > 0\}$;

(iv) sum of positive roots: $2\delta_G = (n-2)\alpha_1 + (n-4)\alpha_2 + \cdots + \alpha_m$;

(v) weights of τ_1: $\pm\alpha_k, 0$; corresponding vectors: $e_{n-2k+1} \pm i\, e_{n-2k+2}$, e_1;

(vi) dominant weight of τ_p $(0 \leqslant p \leqslant m)$: $\mu_p = \alpha_1 + \cdots + \alpha_p$.

- *case* $n = 2m$ *even:*

(i) elements of $\mathfrak{t}_{\mathbb{C}}$ [4]: $T_s = \begin{pmatrix} \begin{pmatrix} 0 & is_m \\ -is_m & 0 \end{pmatrix} & & & \\ & \ddots & & \\ & & \begin{pmatrix} 0 & is_1 \\ -is_1 & 0 \end{pmatrix} & \\ & & & 0 \end{pmatrix}$,

where $s = (s_1, \ldots, s_m) \in \mathbb{C}^m$;

(ii) positive roots: $\alpha_j \pm \alpha_k$ $(j < k)$, where $\alpha_j \in \mathfrak{t}_{\mathbb{C}}^*$ is defined by $\alpha_j(T_s) = s_j$;

(iii) positive Weyl chamber: $\{T_s \in \mathfrak{t}_{\mathbb{C}}^* : s_j \in \mathbb{R},\ s_1 > \ldots > s_{m-1} > |s_m|\}$;

(iv) sum of positive roots: $2\delta_G = (n-2)\alpha_1 + (n-4)\alpha_2 + \cdots + 2\alpha_{m-1}$;

(v) weights of τ_1: $\pm\alpha_k$; corresponding vectors: $e_{n-2k+1} \pm i\, e_{n-2k+2}$;

[4] The unusual looking of these matrices is due to the choice of the parametrization of elements $a_t \in A$ we made in §2.

(vi) dominant weight of τ_p $(0 \leqslant p \leqslant m-1)$: $\mu_p = \alpha_1 + \cdots + \alpha_p$;

(vii) dominant weight of $\tau_m^{\pm}$: $\mu_m^{\pm} = \alpha_1 + \cdots + \alpha_{m-1} \pm \alpha_m$.

***M*-decomposition of τ_p**

Let σ_p be the standard representation of M on $\wedge^p(\mathbb{C}e_2 \oplus \cdots \oplus \mathbb{C}e_n) \simeq \wedge^p\mathbb{C}^{n-1}$. For the decomposition of $\tau_{p|_M}$, we distinguish three cases:

- $p \neq \frac{n-1}{2}, \frac{n}{2}$: the decomposition is obtained using the 'branching law' (see [Boe63], [IT78] or [BS79]):
$$\wedge^p\mathbb{C}^n = e_1 \wedge (\wedge^{p-1}\mathbb{C}^{n-1}) \oplus \wedge^p\mathbb{C}^{n-1},$$
$$\tau_{p|_M} = \sigma_{p-1} \oplus \sigma_p.$$
The factors occuring in the decomposition are irreducible and inequivalent.
- $p = \frac{n-1}{2}$: $\tau_{p|_M} = \sigma_{p-1} \oplus \sigma_p^+ \oplus \sigma_p^-$ is a decomposition into irreducible inequivalent factors ($\sigma_p = \sigma_p^+ \oplus \sigma_p^-$ is the decomposition into eigenspaces for the Hodge operator).
- $p = \frac{n}{2}$: $\tau_{p|_M} = \tau_p^+{}_{|_M} \overset{\sim}{\oplus} \tau_p^-{}_{|_M} = \sigma_{p-1} \overset{\sim}{\oplus} \sigma_p$ are two decompositions into irreducible equivalent factors.

It follows that $\mathrm{Hom}_K(\mathcal{H}_{\sigma,\lambda}, \mathcal{H}_{\tau_p}) \simeq \mathrm{Hom}_M(\mathcal{H}_\sigma, \mathcal{H}_{\tau_p})$ is nontrivial if and only if $\sigma = \sigma_{p-1}, \sigma_p$ (or σ_{p-1} and $\sigma_p^{\pm}$ if $p = \frac{n-1}{2}$). Moreover, this space is one dimensional, except for $p = \frac{n}{2}$ or $p = \frac{n-1}{2}$ and $\sigma = \sigma_p$, in which cases it is two dimensional (but we can always reduce to the multiplicity free case by considering the subrepresentations $\tau_{n/2}^{\pm}$ or $\sigma_{(n-1)/2}^{\pm}$). We now apply these results to the continuous part in (3.4), distinguishing three cases.

(i) *case* $p \neq \frac{n-1}{2}, \frac{n}{2}$

In this case, the continuous part of $L^2(G, K, \tau_p)$ reduces to:
$$\int_{W\backslash(\{\sigma_{p-1},\sigma_p\}\times\mathfrak{a}^*)}^{\oplus} d\nu(\sigma,\lambda)\, \mathcal{H}_{\sigma_q,\lambda}.$$
But we can go a little further: we know that $W = W(\mathfrak{g}, \mathfrak{a})$ can be realized as the quotient M'/M, where $M' \simeq S(\{\pm 1\} \times O(n-1))$ is the normalizer of A in K; thus, $W \simeq \{\pm 1\}$. The nontrivial element $w = m'M$ of W acts on a principal series representation $\pi_{\sigma,\lambda}$ by
$$w \cdot \pi_{\sigma,\lambda} = \pi_{w\cdot\sigma,-\lambda}, \qquad \text{where} \quad w \cdot \sigma(m) = \sigma(m'^{-1}mm'),$$
$$\text{and with} \quad m' = \begin{pmatrix} -I_2 & 0 \\ 0 & I_{n-1} \end{pmatrix}.$$

Since w preserves the highest weights of σ_{p-1} and σ_p, we have here $w\cdot\sigma_q\sim\sigma_q$. This shows that the integral over $W\backslash(\{\sigma_{p-1},\sigma_p\}\times\mathfrak{a}^*)$ can be reduced to an integral over $\{\sigma_{p-1},\sigma_p\}\times\mathbb{R}_+$, and so we get:

$$L^2(G,K,\tau_p)_c\simeq\sum\nolimits^{\oplus}_{q=p-1,p}\int^{\oplus}_{\mathbb{R}_+}d\nu_q(\lambda)\,\mathcal{H}_{\sigma_q,\lambda}. \tag{3.6}$$

(ii) *case* $p=\frac{n-1}{2}$

We adapt the preceding argument to the case of the three representations σ_{p-1}, σ_p^+ and σ_p^-. Looking at the highest weights, one sees that the nontrivial Weyl group element w preserves σ_{p-1} and exchanges σ_p^+ and σ_p^-. Thus the contribution of the series $\pi_{\sigma_p,\lambda}$ reduces to a direct integral over $\{\sigma_p^+,\sigma_p^-\}\times\mathbb{R}_+$, or equivalently over $\{\sigma_p^\pm\}\times\mathbb{R}$. $\pi_{\sigma_p^+,\lambda}$ and $\pi_{\sigma_p^-,-\lambda}$ are indeed equivalent, although σ_p^+ et σ_p^- are not (see [Kna86]: 'Bruhat theory', §VII.3). We finally get:

$$\begin{aligned}L^2(G,K,\tau_p)_c\simeq&\int^{\oplus}_{\mathbb{R}_+}d\nu_{p-1}(\lambda)\,\mathcal{H}_{\sigma_{p-1},\lambda}\\&\oplus\int^{\oplus}_{\mathbb{R}_+}d\nu_p^+(\lambda)\,\mathcal{H}_{\sigma_p^+,\lambda}\oplus\int^{\oplus}_{\mathbb{R}_+}d\nu_p^-(\lambda)\,\mathcal{H}_{\sigma_p^-,\lambda}.\end{aligned} \tag{3.7}$$

(iii) *case* $p=\frac{n}{2}$

We have already noticed the equivalences: $\tau_p^+{}_{|M}\sim\tau_p^-{}_{|M}\sim\sigma_{p-1}\sim\sigma_p$. Thus, one can write:

$$L^2(G,K,\tau_p^\pm)_c\simeq\int^{\oplus}_{\mathbb{R}_+}d\nu_q(\lambda)\,\mathcal{H}_{\sigma_q,\lambda}\qquad(q=p-1\text{ or }p), \tag{3.8}$$

$$\begin{aligned}L^2(G,K,\tau_p)_c&=L^2(G,K,\tau_p^+)_c\oplus L^2(G,K,\tau_p^-)_c\\&\simeq 2\int^{\oplus}_{\mathbb{R}_+}d\nu_q(\lambda)\,\mathcal{H}_{\sigma_q,\lambda}\qquad(q=p-1\text{ or }p).\end{aligned} \tag{3.9}$$

3.3 Discrete series representations decomposing $L^2(G,K,\tau_p)$

In this subsection, we describe the discrete series contribution to $L^2(G,K,\tau_p)$ by specializing general results given in Appendix A. As a matter of fact, our setting ($G/K = H^n(\mathbb{R})$, $p = \frac{n}{2}$), although most simple, provides quite an interesting illustration and served as an important inspiration for the statement of Corollary A.3.

Fix $p = \frac{n}{2}$ and let $\mathfrak{h}_{\mathbb{C}}$ be the compact Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$ whose elements are of the type

$$H_s = \begin{pmatrix} \begin{pmatrix} 0 & is_p \\ -is_p & 0 \end{pmatrix} & & & \\ & \ddots & & \\ & & \begin{pmatrix} 0 & is_1 \\ -is_1 & 0 \end{pmatrix} & \\ & & & 0 \end{pmatrix}, \quad s = (s_1, \dots, s_p) \in \mathbb{C}^p;$$

Here, $R_G = R_K \cup R_{\mathfrak{p}}$, with $R_K = \{\pm\alpha_j \pm \alpha_k, 1 \leqslant j < k \leqslant p\}$ (compact roots) and $R_{\mathfrak{p}} = \{\pm\alpha_j, 1 \leqslant j \leqslant p\}$ (noncompact roots). Then $R_G^+ = \{t_j \pm t_k, j < k\} \cup \{t_j\}$, and the half-sums of positive roots δ_G and δ_K can be written as

$$\delta_G = (p - \tfrac{1}{2})\alpha_1 + (p - \tfrac{3}{2})\alpha_2 + \cdots + \tfrac{3}{2}\alpha_{p-1} + \tfrac{1}{2}\alpha_p,$$
$$\delta_K = (p-1)\alpha_1 + (p-2)\alpha_2 + \cdots + \alpha_{p-1}.$$

If S_p denotes the permutation group of p elements,

$$W_G = \{\pm 1\}^p \rtimes S_p,$$
$$W_K = \{\pm 1\}^p_{even} \rtimes S_p,$$

where $\{\pm 1\}^p_{even} = \{(\varepsilon_1, \dots, \varepsilon_p) : \varepsilon_i = \pm 1, \prod_i \varepsilon_i = +1\}$. Hence,

$$W_G \cdot \delta_G = \{\nu_1\alpha_1 + \cdots + \nu_p\alpha_p : \{|\nu_1|, \dots, |\nu_p|\} = \{\tfrac{1}{2}, \tfrac{3}{2}, \dots, p - \tfrac{1}{2}\}\},$$
$$W_K \backslash W_G \cdot \delta_G = \{W_K \cdot \delta_G, W_K \cdot r_m(\delta_G)\} \quad \text{for any } m \in \{1, \dots, p\},$$

where r_m denotes the reflection $t_m \mapsto -t_m$. Thus we have exactly two discrete series representations in $L^2(G,K,\tau_p)$, with Harish-Chandra parameters $\Lambda^+ = \delta_G$ and $\Lambda^- = r_p(\delta_G)$ (according to the choices $w_1 = \mathrm{Id}$, $w_2 = r_p$ made in Appendix A). We put $\pi^\pm = \pi_{\Lambda^\pm}$.

Since $\tau^{\pm}_{\frac{n}{2}}$ has highest weight $\alpha_1+\cdots+\alpha_{p-1}\pm\alpha_p$, Corollary A.3 yields the following result in our case.

Proposition 3.1. *Let $L^2(G,K,\tau_{\frac{n}{2}})_\Delta$ be the space of L^2 harmonic $\frac{n}{2}$-forms on the real hyperbolic space $H^n(\mathbb{R})$ and $\mathcal{H}_{\pi^\pm}$ be the space of the discrete series representation $\pi^\pm$. Then*

$$L^2(G,K,\tau_{\frac{n}{2}})_\Delta = L^2(G,K,\tau^+_{\frac{n}{2}})_\Delta \oplus L^2(G,K,\tau^-_{\frac{n}{2}})_\Delta,$$

with precisely $L^2(G,K,\tau^{\pm}_{\frac{n}{2}})_\Delta \simeq \mathcal{H}_{\pi^\pm}$.

3.4 The explicit abstract Plancherel Theorem for $L^2(G,K,\tau_p)$

We now have at our disposal all necessary ingredients to make formula (3.4) explicit.

Theorem 3.2 (Abstract Plancherel Theorem). *The space $L^2(G,K,\tau_p)$ of L^2 differential p-forms on the real hyperbolic space $H^n(\mathbb{R})$ decomposes as follows.*

- *For $p \neq \frac{n-1}{2}, \frac{n}{2}$:*

$$L^2(G,K,\tau_p) = \sum\nolimits^{\oplus}_{q=p-1,p} \int^{\oplus}_{\mathbb{R}_+} d\nu_q(\lambda)\,\mathcal{H}_{\sigma_q,\lambda}.$$

- *For $p = \frac{n-1}{2}$:*

$$L^2(G,K,\tau_p) = \int^{\oplus}_{\mathbb{R}_+} d\nu_{p-1}(\lambda)\,\mathcal{H}_{\sigma_{p-1},\lambda} \oplus \int^{\oplus}_{\mathbb{R}_+} d\nu^+_p(\lambda)\,\mathcal{H}_{\sigma^+_p,\lambda} \oplus \int^{\oplus}_{\mathbb{R}_+} d\nu^-_p(\lambda)\,\mathcal{H}_{\sigma^-_p,\lambda}.$$

- *For $p = \frac{n}{2}$:*

$$L^2(G,K,\tau_p) = \mathcal{H}_{\pi^+} \oplus \mathcal{H}_{\pi^-} \oplus 2\int^{\oplus}_{\mathbb{R}_+} d\nu_q(\lambda)\,\mathcal{H}_{\sigma_q,\lambda} \qquad (q = p-1 \text{ or } p),$$

where π^+ and π^- are the discrete series of G with trivial infinitesimal character and constitute the harmonic part of $L^2(G,K,\tau^{\pm}_p)$.

Proof : it suffices to use formulæ (3.6), (3.7), (3.9) and the previous proposition. ✓

Remarks:

1. There exists another method to determine the discrete part of $L^2(G, K, \tau_p)$, which works only for $G/K = H^n(\mathbb{R})$, contrary to Theorem A.2. Klimyk & Gavrilik [KG76, KG79] and Thieleker [Thi73, Thi74, Thi77] have studied the unitary dual of $G = SO_e(n, 1)$. It can be derived from their works that there are exactly two discrete series $\pi^{\pm}$ that occur in the decomposition of $L^2(G, K, \tau_p)$ for $p = n/2$ (and none else), and that they can be embedded into the (nonunitary) principal series representation $\pi_{\sigma,\lambda}$ with parameters $\sigma = \sigma_{\frac{n}{2}-1} \sim \sigma_{\frac{n}{2}}$ and $\lambda = -i/2$ (this is actually the way discrete series arise in our analytic approach to the Plancherel theorem in Section 6) — note that this embedding is also explicitely mentioned in [BS80]. On the other hand, Gaillard has shown in [Gai86] that the sum $\pi^+ \oplus \pi^-$ coincides with $L^2(G, K, \tau_{\frac{n}{2}})_\Delta$. To get the precise identification $\mathcal{H}_{\pi^\pm} = L^2(G, K, \tau^{\pm}_{\frac{n}{2}})_\Delta$, one looks then at the occurrence of the K-type $\tau^{\pm}_{\frac{n}{2}}$ in these representations.

2. Harish-Chandra showed ([HC76b]) that the Plancherel measure $d\nu_\sigma(\lambda)$ is always absolutely continuous with respect to the Lebesgue measure $d\lambda$: $d\nu_\sigma(\lambda) = d\lambda\, \mu_\sigma(\lambda)$, with

$$\mu_\sigma(\lambda)^{-1}\,\mathrm{Id} = \mathrm{cst}\ c_\sigma(\lambda)^*\, c_\sigma(\lambda),$$

for a certain matrix-valued function c_σ. Miatello ([Mia79]) calculated it for real rank-one linear Lie groups. Recall that Hirai ([Hir66]) was actually the first one who obtained the Plancherel formula for $G = SO_e(n, 1)$. Using the latter article, Camporesi & Higuchi ([CH94]) wrote down the Plancherel measure for L^2 p-forms on $H^n(\mathbb{R})$ — see also [Ven93] for a direct calculation of the c_σ-function in the same setting (but only in the case $p \neq \frac{n\pm1}{2}, \frac{n}{2}$). In Section 6 we will deduce the Plancherel measure from the Plancherel formula for the Jacobi transform.

3. For notational reasons, the definition of the Fourier transform for differential forms on $H^n(\mathbb{R})$ is postponed to Section 6.

4 Eigenforms for the algebra $\mathbb{D}(G, K, \tau_p)$

Let π be a representation of G on a Hilbert space $\mathcal{H}_\pi$ whose restriction to K is unitary and contains τ_p with multiplicity one. Consider the orthogonal projection $P_\pi^{\tau_p}$ of $\mathcal{H}_\pi$ onto the isotypic component $\mathcal{H}_{\tau_p}$, which commutes with the action of $\pi_{|_K}$. According to Definition (2.2), for all $v \in \mathcal{H}_\pi$, the function $\phi : G \to \mathcal{H}_{\tau_p}$ defined by

$$\phi(g) = P_\pi^{\tau_p} \circ \pi(g^{-1})v \tag{4.1}$$

is of type τ_p; in other words, it is a p-form on G/K. We shall see that, when π is a representation occuring in the decomposition of $L^2(G, K, \tau_p)$, the form ϕ defined by (4.1) is a Poisson transform of a differential form on the boundary of the hyperbolic space, and, in addition, an eigenform for the algebra $\mathbb{D}(G, K, \tau_p)$.

4.1 Poisson transforms

In §3.2, we have seen that the decomposition of the representation $\tau_{p|_M}$ into (inequivalent) irreducible factors varies with p [1]:

$$\tau_{p|_M} \sim \begin{cases} \sigma_p & \text{if } p = 0, \\ \sigma_{p-1} \oplus \sigma_p & \text{if } 0 < p < \frac{n-1}{2}, \\ \sigma_{p-1} \oplus \sigma_p^+ \oplus \sigma_p^- & \text{if } p = \frac{n-1}{2}, \\ 2\,\sigma_{p-1} \sim 2\,\sigma_p \sim 2\,\tau^\pm_{p\,|_M} & \text{if } p = \frac{n}{2}. \end{cases}$$

As a consequence,

(i) for $p \leqslant \frac{n}{2} - 1$ and $q \leqslant \frac{n-1}{2}$ with $q = p, p+1$: $\tau_q \subset \operatorname{Ind}_M^K \sigma_p$ with multiplicity 1;

(ii) for $p = \frac{n}{2}, \frac{n}{2} - 1$ and $q = \frac{n}{2}$: $\tau_q \subset \operatorname{Ind}_M^K \sigma_p$ with multiplicity 2;

(iii) for $p = q = \frac{n-1}{2}$: $\tau_q \subset \operatorname{Ind}_M^K \sigma_p^\pm$ with multiplicity 1, thus $\tau_q \subset \operatorname{Ind}_M^K \sigma_p$ with multiplicity 2.

Let us realize these embeddings — that we will denote by J_p^q — of τ_q into $\operatorname{Ind}_M^K \sigma_p$. Remember first (see §3.1) that, as a K-module, the space $L^2(G, P, \sigma_p \otimes e^{i\lambda} \otimes 1)$ of the principal series $\pi_{\sigma_p,\lambda}$ is identified to the space $L^2(K, M, \sigma_p)$ of L^2 p-forms on the boundary $G/P \simeq K/M \simeq \mathbb{S}^{n-1}$ of the symmetric space $G/K \simeq H^n(\mathbb{R})$.
In accordance with facts (i), (ii) and (iii) above, we must distinguish several cases, depending on the dimension of the space $\operatorname{Hom}_K(\mathcal{H}_{\tau_q}, L^2(K, M, \sigma_p))$.

[1] As already observed, we can restrict our study to $p \leqslant \frac{n}{2}$ thanks to the Hodge duality.

Proposition 4.1. *For $0 \leqslant p < n$ and $0 \leqslant q \leqslant n$ with $q - p = 0, 1$, denote by P_{σ_p} the projection of $\mathcal{H}_{\tau_q} \simeq \wedge^q \mathbb{C}^n$ onto the isotypic component $\mathcal{H}_{\sigma_p} \simeq \wedge^p \mathbb{C}^{n-1}$, and put:*

$$\begin{aligned} J_p^q &: \mathcal{H}_{\tau_q} \simeq \wedge^q \mathbb{C}^n \longrightarrow \mathcal{H}_{\sigma_p, \lambda|_K} \simeq L^2(K, M, \sigma_p) \\ J_p^q(\xi, \cdot) &= c_{p,q}\, P_{\sigma_p}\{\tau_q(\cdot)^{-1}\xi\}, \end{aligned} \tag{4.2}$$

where $c_{p,q}$ is the constant defined by

$$c_{p,q} = \sqrt{\frac{\dim \tau_q}{\dim \sigma_p}} = \begin{cases} \sqrt{\frac{n}{n-q}} & \text{if } p = q, \\ \sqrt{\frac{n}{q}} & \text{if } p + 1 = q. \end{cases} \tag{4.3}$$

Then:

(i) *J_p^q is an isometric embedding of τ_q into $\operatorname{Ind}_M^K \sigma_p$;*

(ii) *the duality relation*

$$J_p^q(\xi, \cdot) = (-1)^{p(n-q)} * J_{n-p-1}^{n-q}(*\xi, \cdot) \tag{4.4}$$

allows restriction to $p, q \leqslant \frac{n}{2}$;

(iii) *when $p \leqslant \frac{n}{2} - 1$ and $q \leqslant \frac{n-1}{2}$, $\operatorname{Hom}_K(\mathcal{H}_{\tau_q}, L^2(K, M, \sigma_p))$ is one dimensional, and J_p^q is a generator;*

(iv) *when $p = \frac{n}{2}, \frac{n}{2} - 1$ and $q = \frac{n}{2}$, $\operatorname{Hom}_K(\mathcal{H}_{\tau_q}, L^2(K, M, \sigma_p))$ is two dimensional. If $\xi = \xi^+ + \xi^- \in \mathcal{H}_{\tau_q^+} \oplus \mathcal{H}_{\tau_q^-}$, then*

$$J_p^q(\xi, \cdot) = J_p^{q,+}(\xi^+, \cdot) + J_p^{q,-}(\xi^-, \cdot), \tag{4.5}$$

where $J_p^{q,+}$ and $J_p^{q,-}$ are two (orthogonal) isometric K-homomorphisms which embed respectively $\mathcal{H}_{\tau_q^+}$ and $\mathcal{H}_{\tau_q^-}$ into $L^2(K, M, \sigma_p)$, and are defined by

$$J_p^{q,\pm}(\xi^\pm, \cdot) = \sqrt{2}\, P_{\sigma_p}\{\tau_q^\pm(\cdot)^{-1}\xi^\pm\}; \tag{4.6}$$

(v) *when $p = \frac{n-1}{2}$ and $q = \frac{n\pm1}{2}$, $\operatorname{Hom}_K(\mathcal{H}_{\tau_q}, L^2(K, M, \sigma_p))$ is two dimensional, and*

$$J_p^q = \tfrac{1}{\sqrt{2}}(J_{p,+}^q + J_{p,-}^q), \tag{4.7}$$

where $J_{p,\pm}^q$ is an isometric K-homomorphism which embeds $\mathcal{H}_{\tau_q}$ into $L^2(K, M, \sigma_p^\pm)$ and is defined by

$$J_{p,\pm}^q(\xi, \cdot) = 2\sqrt{\tfrac{n}{n+1}}\, P_{\sigma_p^\pm}\{\tau_q(\cdot)^{-1}\xi\}. \tag{4.8}$$

Proof : assertion (i) is trivial; the normalization constants come from Schur's orthogonality relations, and we have $c_{p,q} = \sqrt{C_n^q / C_{n-1}^p}$, where $C_n^q = n!/(q!(n-q)!)$.

(ii) The proof of (4.4) follows from the following observations:

- $c_{n-p-1,n-q} = c_{p,q}$;
- if $\xi \in \wedge^q \mathbb{C}^n$, $\tau_{n-q}(\cdot)^{-1} * \xi = * \tau_q(\cdot)^{-1}\xi$;
- if $\xi \in \wedge^q \mathbb{C}^n$, $* P_{\sigma_{n-p-1}}(*\xi) = (-1)^{p(n-q)} P_{\sigma_p}(\xi)$.

The proof of the third point is short but rather technical. We give it when $q = p$, for example. Recall that the vector $\xi \in \wedge^p \mathbb{C}^n$ can be written as $\xi = e_\alpha + e_1 \wedge e_\beta$, with $e_\alpha \in \wedge^p \mathbb{C}^{n-1}$ and $e_\beta \in \wedge^{p-1}\mathbb{C}^{n-1}$. Both sides of the aimed relation vanish on $*(e_1 \wedge e_\beta)$. On the contrary, if $e_\alpha = e_{i_1} \wedge \ldots \wedge e_{i_p}$ with $2 \leqslant i_1 \leqslant \ldots \leqslant i_p \leqslant n$, then:

$$\begin{aligned} * e_\alpha &= \varepsilon \begin{pmatrix} i_1 & \cdots & i_p & 1 & j_2 & \cdots & j_{n-p} \\ 1 & \cdots & \cdots & \cdots & \cdots & \cdots & n \end{pmatrix} e_1 \wedge e_{j_2} \wedge \ldots \wedge e_{j_{n-p}} \\ &= (-1)^p \, \varepsilon \begin{pmatrix} 1 & i_1 & \cdots & i_p & j_2 & \cdots & j_{n-p} \\ 1 & 2 & \cdots & \cdots & \cdots & \cdots & n \end{pmatrix} e_1 \wedge e_{j_2} \wedge \ldots \wedge e_{j_{n-p}}, \end{aligned}$$

ε being the signature of the considered permutations. Thus,

$$\begin{aligned} * P_{\sigma_{n-p-1}}(* e_\alpha) &= (-1)^p \, \varepsilon \begin{pmatrix} i_1 & \cdots & i_p & j_2 & \cdots & j_{n-p} \\ 2 & \cdots & \cdots & \cdots & \cdots & n \end{pmatrix} *(e_{j_2} \wedge \ldots \wedge e_{j_{n-p}}) \\ &= (-1)^p (-1)^{p(n-p-1)} e_\alpha \\ &= (-1)^{p(n-p)} P_{\sigma_p}(e_\alpha). \end{aligned}$$

Assertion (iii) is clear.

For (iv), one has to take into account the decomposition $\tau_q = \tau_q^+ \oplus \tau_q^-$. Recall that the operator $*$ exchanges isometrically the orthogonal submodules $e_1 \wedge \wedge^{q-1}\mathbb{C}^{n-1}$ and $\wedge^q \mathbb{C}^{n-1}$; let us describe the projectors:

- for q even: $\sqrt{2} P_{\tau_q^\pm} = \frac{\mathrm{Id} \pm *}{\sqrt{2}}$ is an isometry from $e_1 \wedge \wedge^{q-1}\mathbb{C}^{n-1}$ (resp. $\wedge^q \mathbb{C}^{n-1}$) on $\wedge^q_\pm \mathbb{C}^n$. Reciprocally, $\sqrt{2} P_{\sigma_{q-1}}$ (resp. $\sqrt{2} P_{\sigma_q}$) is an isometry from $\wedge^q_\pm \mathbb{C}^n$ on $e_1 \wedge \wedge^{q-1}\mathbb{C}^{n-1}$ (resp. on $\wedge^q \mathbb{C}^{n-1}$);
- for q odd: the same argument holds with $\sqrt{2} P_{\tau_q^\pm} = \frac{\mathrm{Id} \mp i*}{\sqrt{2}}$.

Thus, we can construct the embeddings $J_p^{q,\pm}$ with (4.6), and then J_p^q with (4.5).

In (v), we use the decomposition $\sigma_p = \sigma_p^+ \oplus \sigma_p^-$ to give the definitions (4.8) and (4.7) [2]. Let us point out that $(J_{p,+}^{p+1}, J_{p,-}^{p+1})$ is a basis of $\mathrm{Hom}_K(\mathcal{H}_{\tau_{p+1}}, L^2(K, M, \sigma_p))$.

[2] Notice that, according to the equivalence $\mathrm{Ind}_M^K \sigma_p^+ \sim \mathrm{Ind}_M^K \sigma_p^-$, one has the identity $J_{p,+}^q(\xi, k) = m' J_{p,-}^q(\xi, km')$, where m' is a representative in M' of the nontrivial element of the Weyl group $W(\mathfrak{g}, \mathfrak{a})$ and was defined p. 56.

But, if we define $J_p^{p+1} = (-1)^p {*} J_p^p {*}$ according to (4.4), then $(J_p^{p+1}, J_p^p{*})$ is another basis, which will turn out to be useful for the proof of (4.18) when $p = \frac{n-1}{2}$. ✓

Now, fix $0 \leqslant p \leqslant \frac{n-1}{2}$, $0 \leqslant q \leqslant \frac{n}{2}$ (with $q-p = 0,1$), $\lambda \in \mathbb{C}$, $\omega \in C^\infty(G,P,\sigma_p \otimes e^{i\lambda} \otimes 1)$, and put

$$\phi_p^q(\lambda, g, \omega) = P_p^q \circ \pi_{\sigma_p,\lambda}(g^{-1})\omega, \tag{4.9}$$

$$\phi_p^{q,\pm}(\lambda, g, \omega) = P_p^{q,\pm} \circ \pi_{\sigma_p,\lambda}(g^{-1})\omega, \quad \text{for } q = \tfrac{n}{2}, \tag{4.10}$$

where (in a generic meaning) $P_p^q := (J_p^q)^*$ is the K-equivariant orthogonal projection of $\mathcal{H}_{\sigma_p,\lambda}$ onto $\mathcal{H}_{\tau_q}$ (in other words, it is a generator of $\mathrm{Hom}_K(\mathcal{H}_{\sigma_p,\lambda}, \mathcal{H}_{\tau_q})$). Concretely,

$$P_p^q(\omega) = c_{p,q} \int_K dk\, \tau_q(k)\omega(k).$$

(Note that, using Schur's lemma, $P_p^q \circ J_p^q = \mathrm{Id}$ on $\mathcal{H}_{\tau_q}$.) In the same way, when $p = \frac{n-1}{2}$ and $\omega^\pm \in C^\infty(G,P,\sigma_p^\pm \otimes e^{i\lambda} \otimes 1)$, we put:

$$\phi_{p,\pm}^q(\lambda, g, \omega^\pm) = P_{p,\pm}^q \circ \pi_{\sigma_p^\pm,\lambda}(g^{-1})\omega^\pm. \tag{4.11}$$

Naturally, we have the following relations:

$$\phi_p^q = \phi_p^{q,+} + \phi_p^{q,-}, \quad \text{for } p = \tfrac{n}{2}, \tag{4.12}$$

$$\phi_p^q(\lambda, g, \omega) = \tfrac{1}{\sqrt{2}}\{\phi_{p,+}^q(\lambda, g, \omega^+) + \phi_{p,-}^q(\lambda, g, \omega^-)\}, \tag{4.13}$$
$$\text{for } p = \tfrac{n-1}{2},\ \omega = \omega^+ + \omega^-.$$

Thus, in all cases, $\phi_p^q(\lambda,\cdot,\omega)$ is a C^∞ differential q-form on G/K which is defined as a transform of the C^∞ p-form ω on the boundary G/P of G/K. Since the map

$$\begin{aligned} \phi_p^q(\lambda,\cdot,\cdot) : C^\infty(G,P,\sigma_p \otimes e^{i\lambda} \otimes 1) &\to C^\infty(G,K,\tau_q) \\ \omega &\mapsto \phi_p^q(\lambda,\cdot,\omega) \end{aligned}$$

is continous, linear and G-equivariant, we shall call it *Poisson transform* on $C^\infty(G,P,\sigma_p \otimes e^{i\lambda} \otimes 1)$.

REMARKS:

1. Let us give explicit expressions for the Poisson transforms. For instance, if $p < \frac{n-1}{2}$,

$$\phi_p^q(\lambda, g, \omega) = c_{p,q} \int_K dk\, e^{-(i\lambda+\rho)H(xk)} \tau_q(k)\omega(\underline{k}(xk)).$$

2. The definition of our Poisson transforms coincides, up to the constant $c_{p,q}$, with the ones given in [Olb94], [Ven94] and [Yan94], when applied to our particular bundle.

4.2 Action of $\mathbb{D}(G, K, \tau_p)$ on Poisson transforms

The aim of this section is to show that the Poisson transforms ϕ_p^q (resp. $\phi_{p,\pm}^q$, $\phi_p^{q,\pm}$) defined above are eigenforms for the invariant differential operators of $\mathbb{D}(G, K, \tau_q^{(\pm)})$. For this purpose, we will follow two different approaches: the first one is algebraic, and will give the action of the Casimir operator Ω on the Poisson transforms; the second one is analytic, and will lead to *all* differential equations associated with the invariant differential operators, after having described the action of d and d^* on ϕ_p^q (resp. $\phi_{p,\pm}^q$, $\phi_p^{q,\pm}$). However, intermediate results in the algebraic method are also useful later on, and we develop both aspects of the solution.

The algebraic method

Remind that the *Casimir element* of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ is defined by: $\Omega = \sum g^{ij} X_i X_j$, where (X_i) is any basis of $\mathfrak{g}$ and (g^{ij}) is the inverse of the matrix with coefficients $g_{ij} = B(X_i, X_j)$ (where B denotes the Killing form on $\mathfrak{g}$). In particular, if (Y_i) and $(Z_i)$are the respective standard basis of $\mathfrak{p} \simeq \mathbb{R}^n$ and $\mathfrak{k} = \mathfrak{so}(n)$, one has:

$$B(Y_i, Y_j) = 2(n-1)\delta_{ij}, \quad B(Z_i, Z_j) = -2(n-1)\delta_{ij} \quad \text{and} \quad B(Y_i, Z_j) = 0.$$

In order to recover the standard normalization of the scalar product on $\mathfrak{p}$ corresponding to a sectional curvature equal to -1 for $H^n(\mathbb{R})$, we put $\langle \cdot\,, \cdot \rangle = \frac{1}{2(n-1)} B(\cdot\,, \cdot)$, and we keep the same notation for the Casimir associated with this bilinear form, so that

$$\Omega = \Omega_{\mathfrak{p}} - \Omega_{\mathfrak{k}} = \sum Y_i^2 - \sum Z_j^2. \tag{4.14}$$

Finally, with this choice we have some traditional identifications ('Kuga's formula', see [BW80], Theorem II.2.5(iii)): if, following the notation of Harish-Chandra, we define

$$f(X_1 \ldots X_m : x : Y_1 \ldots Y_n) =$$
$$\frac{\partial}{\partial s_1}\Big|_0 \cdots \frac{\partial}{\partial s_m}\Big|_0 \frac{\partial}{\partial t_1}\Big|_0 \cdots \frac{\partial}{\partial t_n}\Big|_0 f(\exp s_1 X_1 \cdots \exp s_m X_m \cdot x \cdot \exp t_1 Y_1 \cdots \exp t_n Y_n),$$

where $X_1, \ldots, X_m, Y_1, \ldots, Y_n \in \mathfrak{g}$, then for any function $f \in C^\infty(G, K, \tau_p)$:

$$\begin{aligned} f(g : \Omega) = f(\Omega : g) &= -\Delta f(g), \\ f(g : \Omega_{\mathfrak{p}}) &= -\Delta_{\mathfrak{p}} f(g) \quad (\Delta_{\mathfrak{p}} \text{ is } \textit{Bochner's Laplacian}), \\ f(g : \Omega_{\mathfrak{k}}) &= \tau_p(\Omega_{\mathfrak{k}}) f(g), \end{aligned} \tag{4.15}$$

$\Omega_{\mathfrak{k}}$ inducing in this way a constant differential operator of order 0.

Let $0 \leqslant p, q \leqslant n$ such that $q - p = 0, 1$. We want to compute (generically)

$$\phi_p^q(\lambda, g : \Omega, \omega) = P_p^q \circ \pi_{\sigma_p,\lambda}(\Omega) \circ \pi_{\sigma_p,\lambda}(g^{-1})\omega.$$

It is well-known that Ω is a central operator in the enveloping algebra of $\mathfrak{g}$ and that $\pi_{\sigma_p,\lambda}(\Omega)$ is a scalar operator acting by a constant. More precisely, we have (see e.g. [Kna86], Proposition 8.22 and Lemma 12.28):

$$\pi_{\sigma_p,\lambda}(\Omega) = \big\{- \langle\lambda, \lambda\rangle - \langle\rho, \rho\rangle + \langle\mu_{\sigma_p}, \mu_{\sigma_p} + 2\delta_M\rangle\big\} \operatorname{Id},$$

where μ_{σ_p} is the dominant weight of σ_p and δ_M the half-sum of the positive roots in $\mathfrak{m}_{\mathbb{C}}$ with respect to a Cartan subalgebra. Hence:

$$\phi_p^q(\lambda, g : \Omega, \omega) = \{-(\lambda^2 + \rho^2) + \langle\mu_{\sigma_p}, \mu_{\sigma_p} + 2\delta_M\rangle\}\phi_p^q(\lambda, g, \omega). \tag{4.16}$$

Since $M \simeq SO(n-1)$, we can deduce the values of μ_{σ_p} and δ_M from the results we gave for the representation τ_p of $K \simeq SO(n)$ in Paragraph 3.2. They give $\langle\mu_{\sigma_p}, \mu_{\sigma_p} + 2\delta_M\rangle = p(n-1-p)$.
Notice that this last expression is also true for $p = (n \pm 1)/2$ and $\sigma = \sigma_p^{\pm}$; on the other hand, when $q = n/2$, $\phi_p^{q,\pm}$ verifies (4.16) as well. Thus we have proved the following

Proposition 4.2. *The differential forms* $\phi = \phi_p^q(\lambda, \cdot, \omega)$, $\phi_{p,\pm}^q(\lambda, \cdot, \omega)$, $\phi_p^{q,\pm}(\lambda, \cdot, \omega)$ *defined by (4.9), (4.10) and (4.11) verify the differential equation:*

$$\phi(\lambda, g : \Omega, \omega) = -\{\lambda^2 + (\rho - p)^2\}\phi(\lambda, g, \omega).$$

Remark: Kuga's formula (4.15) for p-forms on $H^n(\mathbb{R})$ is in fact a corollary of another famous result, namely *Weitzenböck's formula.* This formula is true for any Riemannian manifold and takes the following form (see e.g. [Poo81], Theorem 4.22): with our notation, for any C^∞ p-form f on the manifold,

$$\Delta f = \Delta_{\mathfrak{p}} f + T_p f,$$

where T_p is an operator of order 0. When the manifold has a constant sectional curvature κ, Weitzenböck's formula becomes ([Poo81], Proposition 4.24):

$$\Delta f = \Delta_{\mathfrak{p}} f + \kappa p(n-p) f.$$

With the same kind of calculations as before, we can quickly find again the formula for the real hyperbolic space, using the action of the Casimir Ω on p-forms and Kuga's formula (4.15): we have

$$f(g : \Omega) = f(g : \Omega_{\mathfrak{p}}) - f(g : \Omega_{\mathfrak{k}}),$$

with

$$\begin{aligned} f(g : \Omega_{\mathfrak{k}}) &= - \langle \mu_{\tau_p}, \mu_{\tau_p} + 2\delta_K \rangle \; f(g) \\ &= -p(n-p) f(g). \end{aligned} \tag{4.17}$$

This is of course still valid when $p = n/2$ and $\tau = \tau_p^{\pm}$, and we can state:

Proposition 4.3 (Weitzenböck's formula). *Let f be a C^∞ differential p-form on the real hyperbolic space $H^n(\mathbb{R})$. Then*

$$f(g : \Omega) = f(g : \Omega_{\mathfrak{p}}) + p(n-p) f(g).$$

(See also [BOS94], Proposition 3.1, for the statement of Weitzenböck's formula in the general setting of vector bundles over a homogeneous space G/H.)

Corollary 4.4 (Comparison of the L^2 spectra of the Hodge-de Rham and Bochner Laplacians). *For $0 \leqslant p \leqslant n$, denote by $\operatorname{spec}\Delta_p$ (resp. $\operatorname{spec}(\Delta_{\mathfrak{p}})_p$) the L^2 spectrum of the Hodge-de Rham (resp. Bochner) Laplacian on p-forms on $H^n(\mathbb{R})$. Then:*

$$\operatorname{spec}\Delta_p = \begin{cases} [(\frac{n-1}{2} - p)^2, +\infty[& \text{if } p \leqslant \frac{n-1}{2}, \\ \{0\} \cup [\frac{1}{4}, +\infty[& \text{if } p = \frac{n}{2}, \\ [(\frac{n+1}{2} - p)^2, +\infty[& \text{if } p \geqslant \frac{n+1}{2}; \end{cases}$$

$$\operatorname{spec}(\Delta_{\mathfrak{p}})_p = \begin{cases} [(\frac{n-1}{2})^2 + p, +\infty[& \text{if } p \leqslant \frac{n-1}{2}, \\ \{\frac{n^2}{4}\} \cup [\frac{n^2+1}{4}, +\infty[& \text{if } p = \frac{n}{2}, \\ [(\frac{n+1}{2})^2 - p, +\infty[& \text{if } p \geqslant \frac{n+1}{2}. \end{cases}$$

In particular, for $0 < p < n$, $\inf \operatorname{spec}(\Delta_{\mathfrak{p}})_p > (\frac{n-1}{2})^2 > \inf \operatorname{spec}\Delta_p$.

Proof : the assertions follow easily from the two previous propositions. The number $(\frac{n-1}{2})^2$ in the final frame corresponds to the value of the infimum of the spectrum of $(\Delta_{\mathfrak{p}})_0 = \Delta_0$. ✓

Remarks:

1. The L^2 spectrum of the Hodge-de Rham Laplacian on forms on the real hyperbolic space had already been computed by Donnelly ([Don81]).

2. The inequality $\inf \operatorname{spec}\Delta_p < \inf \operatorname{spec}\Delta_0 = (\frac{n-1}{2})^2$ results also from the domination of the heat semi-group on $L^2 \wedge^p H^n(\mathbb{R})$ by the heat semi-group on $L^2 H^n(\mathbb{R})$. This result is valid in a rather general context and is usually known as 'Kato's inequality' — see [HSU77] for complements.

The analytic method

The aimed result is the following.

Proposition 4.5. *The differential forms ϕ_p^q, $\phi_p^{q,\pm}$ and $\phi_{p,\pm}^q$ defined respectively by (4.9), (4.10) and (4.11) verify the following differential equations:*

(i) *for $0 \leqslant p \leqslant \frac{n-1}{2}$, $0 \leqslant q \leqslant \frac{n}{2}$ and $q - p = 0, 1$:*

$$d\phi_p^p(\lambda, \cdot, \omega) = \sqrt{\tfrac{p+1}{n-p}}\,(\rho - p - i\lambda)\phi_p^{p+1}(\lambda, \cdot, \omega), \tag{4.18}$$

$$d\phi_p^{p+1}(\lambda, \cdot, \omega) = 0, \tag{4.19}$$

$$d^*\phi_p^p(\lambda, \cdot, \omega) = 0, \tag{4.20}$$

$$d^*\phi_p^{p+1}(\lambda, \cdot, \omega) = \sqrt{\tfrac{n-p}{p+1}}\,(\rho - p + i\lambda)\phi_p^p(\lambda, \cdot, \omega); \tag{4.21}$$

(ii) *in the particular case* $p=\frac{n-1}{2}$*:*

$$d\phi^{p}_{p,\pm}(\lambda,\cdot,\omega)=-i\lambda\,\phi^{p+1}_{p,\pm}(\lambda,\cdot,\omega),\tag{4.22}$$

$$d\phi^{p+1}_{p,\pm}(\lambda,\cdot,\omega)=0,\tag{4.23}$$

$$d^*\phi^{p}_{p,\pm}(\lambda,\cdot,\omega)=0,\tag{4.24}$$

$$d^*\phi^{p+1}_{p,\pm}(\lambda,\cdot,\omega)=i\lambda\,\phi^{p}_{p,\pm}(\lambda,\cdot,\omega);\tag{4.25}$$

(iii) *in the particular case* $q=\frac{n}{2}$*:*

$$d\phi^{q,\pm}_{q}(\lambda,\cdot,\omega)=-\tfrac12\sqrt{\tfrac{n+2}{n}}(\tfrac12+i\lambda)\phi^{q+1}_{q}(\lambda,\cdot,\omega),\tag{4.26}$$

$$d\phi^{q,\pm}_{q-1}(\lambda,\cdot,\omega)=\pm\tfrac12 i^{3q^2+2}\sqrt{\tfrac{n+2}{n}}(\tfrac12+i\lambda)\phi^{q+1}_{q}(\lambda,\cdot,*\omega)\tag{4.27}$$

$$d^*\phi^{q,\pm}_{q}(\lambda,\cdot,\omega)=\pm\tfrac12 i^{3q^2}\sqrt{\tfrac{n+2}{n}}(\tfrac12+i\lambda)\phi^{q-1}_{q-1}(\lambda,\cdot,*\omega),\tag{4.28}$$

$$d^*\phi^{q,\pm}_{q-1}(\lambda,\cdot,\omega)=\tfrac12\sqrt{\tfrac{n+2}{n}}(\tfrac12+i\lambda)\phi^{q-1}_{q-1}(\lambda,\cdot,\omega).\tag{4.29}$$

The following lemma will allow us to show only half of the results of the previous proposition.

Lemma 4.6 (Duality). *The operator $*$ acts on the differential forms ϕ^q_p, $\phi^{q,\pm}_p$ and $\phi^q_{p,\pm}$ in the following way:*

(i) *for* $0\leqslant p\leqslant\frac{n-1}{2}$, $0\leqslant q\leqslant\frac{n}{2}$ *and* $q-p=0,1$*:*

$$*\phi^{q}_{p}(\lambda,\cdot,\omega)=(-1)^{pq}\phi^{n-q}_{n-p-1}(\lambda,\cdot,*\omega);\tag{4.30}$$

(ii) *in the particular case* $p=\frac{n-1}{2}$*:*

$$*\phi^{q}_{p,\pm}(\lambda,\cdot,\omega)=\pm i^{p(p+2q)}\phi^{n-q}_{p,\pm}(\lambda,\cdot,\omega);\tag{4.31}$$

(iii) *in the particular case* $q=\frac{n}{2}$*:*

$$*\phi^{q,\pm}_{p}(\lambda,\cdot,\omega)=(-1)^{pq}\phi^{q,\pm}_{n-p-1}(\lambda,\cdot,*\omega);\tag{4.32}$$

Moreover, the relations (4.30), (4.31), (4.32) are verified for $q=p$ if and only if they are verified for $q=p+1$.

Proof of Lemma 4.6: let us show that if (4.30) is true for $q=p$, then (4.30) is true for $q=p+1$. If $p'=n-p-1$,

$$\begin{aligned}*\phi^{p+1}_{p}(\lambda,\cdot,\omega)&=(-1)^{pp'}*\phi^{n-p'}_{n-p'-1}(\lambda,\cdot,**\omega)\\&=(-1)^{(p+1)p'}**\phi^{p'}_{p'}(\lambda,\cdot,*\omega)\\&=(-1)^{(p+1)p'}(-1)^{p'(n-p')}\phi^{p'}_{p'}(\lambda,\cdot,*\omega)\\&=\phi^{p'}_{p'}(\lambda,\cdot,*\omega).\end{aligned}$$

Reciprocally:

$$\begin{aligned} *\phi_p^p(\lambda,\cdot,\omega) &= (-1)^{pp'} *\phi_p^p(\lambda,\cdot,**\omega) \\ &= (-1)^{pp'} **\phi_{n-p-1}^{n-p}(\lambda,\cdot,*\omega) \\ &= (-1)^p \phi_{n-p-1}^{n-p}(\lambda,\cdot,*\omega). \end{aligned}$$

To show (4.30), it suffices now to prove (4.30) for $q = p$ (for instance). Let $\xi \in \wedge^{n-p}\mathbb{C}^n$. On the one hand,

$$\begin{aligned} \big(*\phi_p^p(\lambda,g,\omega),\xi\big)_{\wedge^{n-p}\mathbb{C}^n} &= \big(**\phi_p^p(\lambda,g,\omega),*\xi\big)_{\wedge^p\mathbb{C}^n} \\ &= (-1)^{p(n-p)} \big(P_p^p \circ \pi_{\sigma_p,\lambda}(g^{-1})\omega,*\xi\big)_{\wedge^p\mathbb{C}^n} \\ &= (-1)^{p(n-p)} \big(\pi_{\sigma_p,\lambda}(g^{-1})\omega, J_p^p(*\xi,\cdot)\big)_{L^2(K,M,\sigma_p)} . \end{aligned}$$

On the other hand,

$$\begin{aligned} &\big(\phi_{n-p-1}^{n-p}(\lambda,g,*\omega),\xi\big)_{\wedge^{n-p}\mathbb{C}^n} \\ &= \big(P_{n-p-1}^{n-p} \circ \pi_{\sigma_{n-p-1},\lambda}(g^{-1})*\omega,\xi\big)_{\wedge^{n-p}\mathbb{C}^n} \\ &= \big(\pi_{\sigma_{n-p-1},\lambda}(g^{-1})*\omega, J_{n-p-1}^{n-p}(\xi,\cdot)\big)_{L^2(K,M,\sigma_{n-p-1})} . \end{aligned}$$

Consider the operator

$$\begin{aligned} E : L^2(K,M,\sigma_{n-p-1}) &\longrightarrow L^2(K,M,\sigma_p) \\ \omega &\longmapsto *\omega \end{aligned}$$

- E is well-defined: $*\omega(km) = *\sigma_{n-p-1}(m^{-1})\omega(k) = \sigma_p(m^{-1})*\omega(k)$;
- E is an isometry: $|*\omega(k)| = |\omega(k)|$;
- E intertwines $\pi_{\sigma_{n-p-1},\lambda}$ with $\pi_{\sigma_p,\lambda}$:

$$*\{\pi_{\sigma_{n-p-1},\lambda}(g)\omega\}(k) = e^{-(i\lambda+\rho)H(g^{-1}k)} *\omega(\underline{k}(g^{-1}k)) = \{\pi_{\sigma_p,\lambda}(g)*\omega\}(k).$$

It suffices then to use the duality formula (4.4) to complete the proof of (4.30). The proofs of (4.31) and (4.32) can be done exactly in the same manner; for (4.31), one uses the additional identity $*\xi^{\pm} = \pm i^{p(p+2)}\xi^{\pm}$ for any vector $\xi^{\pm} \in \wedge_{\pm}^{p}\mathbb{C}^{n-1}$ when $p = (n-1)/2$ (see (3.5)). ✓

Proof of Proposition 4.5:

We start by noting that, thanks to the previous lemma, equations (4.18), (4.19), (4.22), (4.23), (4.26) and (4.27) are respectively equivalent to equations (4.21), (4.20), (4.25), (4.24), (4.29) and (4.28).

For example, let us show (4.19) $\Rightarrow$ (4.20) (the reciprocal implication is obtained in the same way).

$$\begin{aligned} d^*\phi_p^p(\lambda, \cdot, \omega) &= (-1)^{n(p+1)+1} {*d*}\phi_p^p(\lambda, \cdot, \omega) \\ &= (-1)^{(n+1)(p+1)} {*d}\phi_{n-p-1}^{n-p}(\lambda, \cdot, *\omega) \\ &= 0. \end{aligned}$$

Similarly (4.18) $\Rightarrow$ (4.21) (the proof of the reciprocal is also omitted).

$$\begin{aligned} d^*\phi_p^{p+1}(\lambda, \cdot, \omega) &= (-1)^{np+1} {*d*}\phi_p^{p+1}(\lambda, \cdot, \omega) \\ &= (-1)^{np+1} {*d}\phi_{n-p-1}^{n-p-1}(\lambda, \cdot, *\omega) \\ &= (-1)^{np+1} \sqrt{\tfrac{n-p}{p+1}}(\rho - n + p + 1 - i\lambda) {*\phi}_{n-p-1}^{n-p}(\lambda, \cdot, *\omega) \\ &= (-1)^{np} \sqrt{\tfrac{n-p}{p+1}}\,(\rho - p + i\lambda)\phi_p^p(\lambda, \cdot, **\omega) \\ &= \sqrt{\tfrac{n-p}{p+1}}\,(\rho - p + i\lambda)\phi_p^p(\lambda, \cdot, \omega). \end{aligned}$$

The other equivalences are shown using the same kind of calculations, with the particularity, however, that all equations of case (iii) are equivalent, because of (3.5).

Before to prove the non-equivalent equations case by case, we recall the expressions of the differential and codifferential maps on forms on the hyperbolic space.
Denote by $(e_j)_{j=1}^n$ the canonical orthonormal basis of $\mathfrak{p} \simeq \mathbb{R}^n$ and by $(\varepsilon_j)_{j=1}^n$ the dual basis of $\mathfrak{p}^*$. Then, for any p and any $f \in C^\infty(G, K, \tau_p)$:

$$\begin{aligned} df(g) &= \sum_{j=1}^n \varepsilon_j \wedge f(g : e_j), \\ d^*f(g) &= -\sum_{j=1}^n i_{e_j} f(g : e_j), \end{aligned}$$

where i_{e_j} is the interior product by the vector e_j.

REMARK: in the sequel, thanks to the duality lemma, we will not need the expression of d^*.

• *Proof of (4.19) in the cases* $0 \leqslant p < n$, $p \neq \frac{n}{2}-1, \frac{n-3}{2}$

$$\begin{aligned}
\big(d\phi_p^{p+1}(\lambda,g,\omega),\xi\big)_{\wedge^{p+2}\mathbb{C}^n} &= \sum_{j=1}^n \big(e_j \wedge \phi_p^{p+1}(g:e_j),\xi\big)_{\wedge^{p+2}\mathbb{C}^n} \\
&= \sum_{j=1}^n \big(\phi_p^{p+1}(g:e_j), i_{e_j}\xi\big)_{\wedge^{p+1}\mathbb{C}^n} \\
&= \sum_{j=1}^n \big(P_p^{p+1}\circ \pi_{\sigma_p,\lambda}(-e_j)\circ \pi_{\sigma_p,\lambda}(g^{-1})\omega, i_{e_j}\xi\big)_{\wedge^{p+1}\mathbb{C}^n} \\
&= \sum_{j=1}^n \big(\pi_{\sigma_p,\lambda}(g^{-1})\omega, \pi_{\sigma_p,\bar\lambda}(e_j)\circ J_p^{p+1}(i_{e_j}\xi,\cdot)\big)_{L^2(K,M,\sigma_p)} \\
&= \big(\pi_{\sigma_p,\lambda}(g^{-1})\omega, T\xi\big)_{L^2(K,M,\sigma_p)},
\end{aligned}$$

where $T:\xi\mapsto \sum_j \pi_{\sigma_p,\bar\lambda}(e_j)\circ J_p^{p+1}(i_{e_j}\xi,\cdot)$ is a K-homomorphism from $\wedge^{p+2}\mathbb{C}^n$ to $L^2(K,M,\sigma_p)$. Indeed,

$$\begin{aligned}
\pi_{\sigma_p,\bar\lambda}(k)\circ T\circ \tau_{p+2}(k^{-1})\xi &= \pi_{\sigma_p,\bar\lambda}(k)\circ \sum_j \pi_{\sigma_p,\bar\lambda}(e_j)\circ J_p^{p+1}(i_{e_j}(\tau_{p+2}(k^{-1})\xi),\cdot) \\
&= \pi_{\sigma_p,\bar\lambda}(k)\circ \sum_j \pi_{\sigma_p,\bar\lambda}(e_j)\circ J_p^{p+1}(\tau_{p+1}(k^{-1})\{i_{\tau_1(k)e_j}\xi\},\cdot) \\
&= \sum_j \pi_{\sigma_p,\bar\lambda}(k)\circ \pi_{\sigma_p,\bar\lambda}(e_j)\circ \pi_{\sigma_p,\bar\lambda}(k^{-1})\circ J_p^{p+1}(i_{\tau_1(k)e_j}\xi,\cdot) \\
&= \sum_j \pi_{\sigma_p,\bar\lambda}(\tau_1(k)e_j)\circ J_p^{p+1}(i_{\tau_1(k)e_j}\xi,\cdot). \qquad (4.33)
\end{aligned}$$

The definition of $T\xi$ being independent of the orthonormal basis (e_j) considered, we can replace $\tau_1(k)e_j$ by $\sum_l \tau_1(k)_{l_j}e_l$; the right-hand side of (4.33) is then equal to

$$\sum_{l=1}^n \pi_{\sigma_p,\bar\lambda}(e_l)\circ J_p^{p+1}(i_{e_l}\xi,\cdot) = T\xi.$$

Now, $\mathrm{Hom}_K(\wedge^{p+2}\mathbb{C}^n, L^2(K,M,\sigma_p)) = \{0\}$, so $T\equiv 0$.

• *Proof of (4.18) in the cases* $0 \leqslant p < n$, $p \neq \frac{n}{2}-1, \frac{n-1}{2}$

$$
\begin{aligned}
\big(d\phi_p^p(\lambda,g,\omega),\xi\big)_{\wedge^{p+1}\mathbb{C}^n} &= \sum_{j=1}^n \big(e_j \wedge \phi_p^p(g:e_j),\xi\big)_{\wedge^{p+1}\mathbb{C}^n}\\
&= \sum_{j=1}^n \big(\phi_p^p(g:e_j), i_{e_j}\xi\big)_{\wedge^{p}\mathbb{C}^n}\\
&= \sum_{j=1}^n \big(P_p^p \circ \pi_{\sigma_p,\lambda}(-e_j)\circ \pi_{\sigma_p,\lambda}(g^{-1})\omega, i_{e_j}\xi\big)_{\wedge^{p}\mathbb{C}^n}\\
&= \sum_{j=1}^n \big(\pi_{\sigma_p,\lambda}(g^{-1})\omega, \pi_{\sigma_p,\bar\lambda}(e_j)\circ J_p^p(i_{e_j}\xi,\cdot)\big)_{L^2(K,M,\sigma_p)}\\
&=\big(\pi_{\sigma_p,\lambda}(g^{-1})\omega, T\xi\big)_{L^2(K,M,\sigma_p)},
\end{aligned}
$$

where $T:\xi\mapsto \sum_j \pi_{\sigma_p,\bar\lambda}(e_j)\circ J_p^p(i_{e_j}\xi,\cdot)$ is a K-homomorphism from $\wedge^{p+1}\mathbb{C}^n$ to $L^2(K,M,\sigma_p)$ — this can be shown as in the proof of (4.19). Here, $\mathrm{Hom}_K(\wedge^{p+1}\mathbb{C}^n, L^2(K,M,\sigma_p))$ is one dimensional and generated by J_p^{p+1}. Thus, $T = c\,J_p^{p+1}$, where c is a constant to be determined. In this view, we evaluate the expressions of T and J_p^{p+1} on $\xi = e_1\wedge\ldots\wedge e_{p+1}$ and $k=e$. On the one hand,

$$
\begin{aligned}
T\xi(e) = \sqrt{\tfrac{n}{n-p}}\sum_{j=1}^{p+1}(-1)^{j-1}\frac{d}{dt}\Big|_{t=0} & e^{-(i\bar\lambda+\rho)H(\exp -te_j)}\\
&\times P_{\sigma_p}\{\tau_p(\underline{k}(\exp -te_j)^{-1})e_1\wedge\ldots\wedge\widehat{e_j}\wedge\ldots\wedge e_{p+1}\}.
\end{aligned}
$$

∘ $\frac{d}{dt}\Big|_{t=0} H(\exp -te_j)$ is the projection of $-e_j$ onto $\mathfrak{a}$ along $\mathfrak{k}\oplus\mathfrak{n}$, so

$$
\frac{d}{dt}\Big|_{t=0} H(\exp -te_j) = \begin{cases} -e_1 & \text{if } j=1,\\ 0 & \text{if } j>1;\end{cases}
$$

∘ $\frac{d}{dt}\Big|_{t=0} \underline{k}(\exp -te_j)^{-1}$ is the projection of e_j onto $\mathfrak{k}$ along $\mathfrak{a}\oplus\mathfrak{n}$, so

$$
\frac{d}{dt}\Big|_{t=0} \underline{k}(\exp -te_j)^{-1} = \begin{cases} 0 & \text{if } j=1,\\ \begin{pmatrix} 0 & \ldots & -1 & \ldots & 0\\ \vdots & & \vdots & & \vdots\\ 1 & \ldots & 0 & \ldots & 0\\ \vdots & & \vdots & & \vdots\\ 0 & \ldots & 0 & \ldots & 0\end{pmatrix} \begin{matrix} \\ \\ \leftarrow j \\ \\ \\ \end{matrix} \\ \qquad\qquad \uparrow j & \text{if } j>1.\end{cases}
$$

To understand this last fact, remind that if one defines the subspaces $\mathfrak{q}\subset\mathfrak{p}$, $\mathfrak{l}\subset\mathfrak{k}$ and $\bar{\mathfrak{n}}\subset\mathfrak{g}$ by $\mathfrak{p}=\mathfrak{q}\overset{\perp}{\oplus}\mathfrak{a}$, $\mathfrak{k}=\mathfrak{l}\overset{\perp}{\oplus}\mathfrak{m}$ and $\bar{\mathfrak{n}}=\theta(\mathfrak{n})$, one has $\mathfrak{l}\oplus\mathfrak{q}=\mathfrak{n}\oplus\bar{\mathfrak{n}}$ [3], so that $e_j\in\mathfrak{q}$ decomposes in a unique way into the sum of an element of $\mathfrak{n}$ and of an element of $\mathfrak{l}$:

$$e_j=\begin{pmatrix}0&\dots&0&\dots&0\\ \vdots&&\vdots&&\vdots\\ 0&\dots&0&\dots&1\\ \vdots&&\vdots&&\vdots\\ 0&\dots&1&\dots&0\end{pmatrix}=\begin{pmatrix}0&\dots&1&\dots&0\\ \vdots&&\vdots&&\vdots\\ -1&\dots&0&\dots&1\\ \vdots&&\vdots&&\vdots\\ 0&\dots&1&\dots&0\end{pmatrix}+\begin{pmatrix}0&\dots&-1&\dots&0\\ \vdots&&\vdots&&\vdots\\ 1&\dots&0&\dots&0\\ \vdots&&\vdots&&\vdots\\ 0&\dots&0&\dots&0\end{pmatrix}.$$

Hence,

$$\begin{aligned}T\xi(e)=&\sqrt{\tfrac{n}{n-p}}\Big\{(i\bar{\lambda}+\rho)P_{\sigma_p}(e_2\wedge\ldots\wedge e_{p+1})\\ &\qquad+\sum_{j=2}^{p+1}(-1)^{j-1}P_{\sigma_p}(e_j\wedge e_2\wedge\ldots\wedge\widehat{e_j}\wedge\ldots\wedge e_{p+1})\Big\}\\ =&\sqrt{\tfrac{n}{n-p}}\Big\{(i\bar{\lambda}+\rho)e_2\wedge\ldots\wedge e_{p+1}-\sum_{j=2}^{p+1}e_2\wedge\ldots\wedge e_{p+1}\Big\}\\ =&\sqrt{\tfrac{n}{n-p}}(i\bar{\lambda}+\rho-p)e_2\wedge\ldots\wedge e_{p+1}.\end{aligned}$$

On the other hand, $J_p^{p+1}(\xi,e)=\sqrt{\frac{n}{p+1}}\,e_2\wedge\ldots\wedge e_{p+1}$. This implies that:

$$c=\sqrt{\tfrac{p+1}{n-p}}(i\bar{\lambda}+\rho-p).$$

Finally,

$$\begin{aligned}\big(d\phi_p^p(\lambda,g,\omega),\xi\big)_{\wedge^{p+1}\mathbb{C}^n}=&\big(\pi_{\sigma_p,\lambda}(g^{-1})\omega,T\xi\big)_{L^2(K,M,\sigma_p)}\\ =&\sqrt{\tfrac{p+1}{n-p}}(\rho-p-i\lambda)\ \big(\pi_{\sigma_p,\lambda}(g^{-1})\omega,J_p^{p+1}\xi\big)_{L^2(K,M,\sigma_p)}\\ =&\sqrt{\tfrac{p+1}{n-p}}(\rho-p-i\lambda)\ \big(P_p^{p+1}\circ\pi_{\sigma_p,\lambda}(g^{-1})\omega,\xi\big)_{\wedge^{p+1}\mathbb{C}^n}\\ =&\Big(\sqrt{\tfrac{p+1}{n-p}}(\rho-p-i\lambda)\phi_p^{p+1}(\lambda,g,\omega),\xi\Big)_{\wedge^{p+1}\mathbb{C}^n}.\end{aligned}$$

- *Proof of (4.19) in the case n even, $p=\frac{n}{2}-1$*

[3] In fact, the orthogonal complement of $\mathfrak{m}\oplus\mathfrak{a}$ in $\mathfrak{g}$ is the direct sum of any two factors choosed among $\mathfrak{n}$, $\bar{\mathfrak{n}}$, $\mathfrak{q}$ and $\mathfrak{l}$.

As before, we have:

$$\big(d\phi_p^{p+1}(\lambda,g,\omega),\xi\big)_{\wedge^{p+2}\mathbb{C}^n}=\cdots=\big(\pi_{\sigma_p,\lambda}(g^{-1})\omega,T\xi\big)_{L^2(K,M,\sigma_p)},$$

where $T:\xi\mapsto\sum_j\pi_{\sigma_p,\bar\lambda}(e_j)\circ J_p^{p+1}(i_{e_j}\xi,\cdot)$ is a K-homomorphism from $\wedge^{p+2}\mathbb{C}^n$ to $L^2(K,M,\sigma_p)$. But, here, $\mathrm{Hom}_K(\wedge^{p+2}\mathbb{C}^n,L^2(K,M,\sigma_p))=\mathbb{C}J_p^p*$, so $T=c\,J_p^p*$ for a constant c to be determined. Let us evaluate this equality on $\xi=e_1\wedge\ldots\wedge e_{p+2}$ and on $k=e$:

$$\begin{aligned}T\xi(e)&=\sqrt2\sum_{j=1}^{p+2}(-1)^{j-1}\frac{d}{dt}\Big|_{t=0}e^{-(i\bar\lambda+\rho)H(\exp -te_j)}\\&\qquad\times P_{\sigma_p}\{\tau_{p+1}(\underline{k}(\exp -te_j))^{-1}e_1\wedge\ldots\wedge\widehat{e_j}\wedge\ldots\wedge e_{p+2}\}\\&=\sqrt2\Big\{(i\bar\lambda+\rho)P_{\sigma_p}(e_2\wedge\ldots\wedge e_{p+2})-\sum_{j=2}^{p+2}P_{\sigma_p}(e_2\wedge\ldots\wedge e_{p+2})\Big\}\\&=0.\end{aligned}$$

But $J_p^p(*\xi,e)=\sqrt{\frac{2n}{n+2}}e_{p+3}\wedge\ldots\wedge e_n$, so that $c=0$, and $d\phi_p^{p+1}=0$.

- *Proof of (4.19) in the case n odd, $p=\frac{n-3}{2}$*

$$\big(d\phi_p^{p+1}(\lambda,g,\omega),\xi\big)_{\wedge^{p+2}\mathbb{C}^n}=\cdots=\big(\pi_{\sigma_p,\lambda}(g^{-1})\omega,T\xi\big)_{L^2(K,M,\sigma_p)},$$

where $T:\xi\mapsto\sum_j\pi_{\sigma_p,\bar\lambda}(e_j)\circ J_p^{p+1}(i_{e_j}\xi,\cdot)$ is a K-homomorphism from $\wedge^{p+2}\mathbb{C}^n$ to $L^2(K,M,\sigma_p)$. Since $\mathrm{Hom}_K(\wedge^{p+2}\mathbb{C}^n,L^2(K,M,\sigma_p))=\mathbb{C}J_p^{p+1}*$, $T=c\,J_p^{p+1}*$, for a constant c to be precised. If $\xi=e_{p+2}\wedge\ldots\wedge e_n$ and $k=e$, one shows that:

$$\begin{aligned}T\xi(e)&=0,\\J_p^{p+1}(*\xi,e)&=\sqrt{\tfrac{2n}{n-1}}\,e_2\wedge\ldots\wedge e_{p+1},\end{aligned}$$

hence $c=0$ and $d\phi_p^{p+1}=0$.

- *Proof of (4.18) in the case n even, $p=\frac{n}{2}-1$*

$$\big(d\phi_p^{p}(\lambda,g,\omega),\xi\big)_{\wedge^{p+1}\mathbb{C}^n}=\cdots=\big(\pi_{\sigma_p,\lambda}(g^{-1})\omega,T\xi\big)_{L^2(K,M,\sigma_p)},$$

where $T:\xi\mapsto\sum_j\pi_{\sigma_p,\bar\lambda}(e_j)\circ J_p^{p}(i_{e_j}\xi,\cdot)$ is a K-homomorphism from $\wedge^{p+1}\mathbb{C}^n$ to $L^2(K,M,\sigma_p)$. Here, $\mathrm{Hom}_K(\wedge^{p+1}\mathbb{C}^n,L^2(K,M,\sigma_p))$ is two dimensional, and generated by J_p^{p+1} and J_{p+1}^{p+1}. So $T=c_1J_p^{p+1}+c_2J_{p+1}^{p+1}$, for some constants c_1 and c_2. We first evaluate this equality on $\xi = e_1 \wedge \ldots \wedge e_{p+1}$ and on $k = e$. We find:

$$\begin{aligned} T\xi(e) &= \sqrt{\tfrac{2n}{n+2}}(\tfrac{1}{2} + i\bar{\lambda})e_2 \wedge \ldots \wedge e_{p+1}, \\ J_p^{p+1}(\xi, e) &= \sqrt{2}\, e_2 \wedge \ldots \wedge e_{p+1}, \\ \text{and } J_{p+1}^{p+1}(\xi, e) &= 0, \end{aligned}$$

from which we deduce $c_1 = \sqrt{\frac{n}{n+2}}(\frac{1}{2} + i\bar{\lambda})$. We now also evaluate T on $\xi = e_{j_1} \wedge \ldots \wedge e_{j_{p+1}}$, with $1 < j_1 < \ldots < j_{p+1} \leqslant n$, and on $k = e$:

$$\begin{aligned} T\xi(e) &= 0, \\ J_p^{p+1}(\xi, e) &= 0, \\ \text{and } *J_{p+1}^{p+1}(\xi, e) &= \sqrt{2}*\xi, \end{aligned}$$

so that we get $c_2 = 0$. It follows that $T = \sqrt{\frac{n}{n+2}}(\frac{1}{2} + i\bar{\lambda})J_p^{p+1}$, and we conclude as before.

- *Proof of (4.18) in the case n odd, $p = \frac{n-1}{2}$*

$$\big(d\phi_p^p(\lambda, g, \omega), \xi\big)_{\wedge^{p+1}\mathbb{C}^n} = \cdots = \big(\pi_{\sigma_p,\lambda}(g^{-1})\omega, T\xi\big)_{L^2(K,M,\sigma_p)},$$

where $T : \xi \mapsto \sum_j \pi_{\sigma_p,\bar{\lambda}}(e_j) \circ J_p^p(i_{e_j}\xi, \cdot)$ is a K-homomorphism from $\wedge^{p+1}\mathbb{C}^n$ to $L^2(K, M, \sigma_p)$. Here, $\mathrm{Hom}_K(\wedge^{p+1}\mathbb{C}^n, L^2(K, M, \sigma_p))$ is two dimensional, generated by J_p^{p+1} and J_p^p*. Thus $T = c_1 J_p^{p+1} + c_2 J_p^p*$, for some constants c_1 et c_2. Evaluating on $\xi = e_1 \wedge \ldots \wedge e_{p+1}$ and $k = e$, we get:

$$\begin{aligned} T\xi(e) &= \sqrt{\tfrac{2n}{n-1}}\, i\bar{\lambda}\, e_2 \wedge \ldots \wedge e_{p+1}, \\ J_p^{\frac{n+1}{2}}(\xi, e) &= \sqrt{\tfrac{2n}{n-1}}\, e_2 \wedge \ldots \wedge e_{p+1}, \\ \text{and } J_p^p(*\xi, e) &= \sqrt{\tfrac{2n}{n-1}}\, e_{p+2} \wedge \ldots \wedge e_n, \end{aligned}$$

so that $c_1 = i\bar{\lambda}$ and $c_2 = 0$. The conclusion is immediate.

The calculations we just carried out prove equations (4.18) and (4.19) in case (i), and easily adapt to cases (ii) and (iii) to obtain (4.22), (4.23) and (4.26). As remarked previously, this suffices to complete the proof of Proposition 4.5. ✓

Corollary 4.7 (Action of invariant differential operators on Poisson transforms). *For a given $q \leqslant n/2$, let us describe the action of the generators of the algebra $\mathbb{D}(G, K, \tau_q)$ on Poisson transforms of degree q:*

(i) *if $q < \frac{n-1}{2}$:*

$$dd^*\phi_q^q(\lambda, \cdot, \omega) = 0, \tag{4.34}$$

$$d^*d\phi_q^q(\lambda, \cdot, \omega) = \{\lambda^2 + (\rho - q)^2\}\phi_q^q(\lambda, \cdot, \omega), \tag{4.35}$$

$$dd^*\phi_{q-1}^q(\lambda, \cdot, \omega) = \{\lambda^2 + (\rho - q + 1)^2\}\phi_{q-1}^q(\lambda, \cdot, \omega), \tag{4.36}$$

$$d^*d\phi_{q-1}^q(\lambda, \cdot, \omega) = 0; \tag{4.37}$$

(ii) *if $q = \frac{n-1}{2}$:*

$$*d\phi_{q,\pm}^q(\lambda, \cdot, \omega) = \pm i^{q^2-1}\lambda\, \phi_{q,\pm}^q(\lambda, \cdot, \omega), \tag{4.38}$$

$$dd^*\phi_{q,\pm}^q(\lambda, \cdot, \omega) = 0, \tag{4.39}$$

$$*d\phi_{q-1}^q(\lambda, \cdot, \omega) = 0, \tag{4.40}$$

$$dd^*\phi_{q-1}^q(\lambda, \cdot, \omega) = (\lambda^2 + 1)\phi_{q-1}^q(\lambda, \cdot, \omega); \tag{4.41}$$

(iii) *if $q = \frac{n}{2}$:*

$$\Delta\phi_p^{q,\pm}(\lambda, \cdot, \omega) = (\lambda^2 + \tfrac{1}{4})\phi_p^{q,\pm}(\lambda, \cdot, \omega) \quad (p = q - 1, q), \tag{4.42}$$

$$*\phi_q^q(\lambda, \cdot, \omega) = (-1)^q\phi_{q-1}^q(\lambda, \cdot, *\omega), \tag{4.43}$$

$$d{*}d\phi_q^q(\lambda, \cdot, \omega) = -(\lambda^2 + \tfrac{1}{4})\phi_{q-1}^q(\lambda, \cdot, *\omega), \tag{4.44}$$

$$*\phi_{q-1}^q(\lambda, \cdot, \omega) = \phi_q^q(\lambda, \cdot, *\omega), \tag{4.45}$$

$$d{*}d\phi_{q-1}^q(\lambda, \cdot, \omega) = 0. \tag{4.46}$$

In particular, each Poisson transform $\phi = \phi_p^q, \phi_p^{q,\pm}, \phi_{p,\pm}^p$ verifies the following differential equation:

$$\Delta\phi(\lambda, \cdot, \omega) = \{\lambda^2 + (\rho - p)^2\}\phi(\lambda, \cdot, \omega). \tag{4.47}$$

In this way, we find again the result obtained in Proposition 4.2.

Proof : remind that the generators of the algebra $\mathbb{D}(G, K, \tau_p)$ had been described in Corollary 2.2. To get the differential equations above, it then suffices to use

Proposition 4.5. We give the detail of the calculation in the most technical case, that is to show (4.42).
On the one hand, according to (4.26) and (4.21),

$$\begin{aligned} d^* d\phi_q^{q,\pm}(\lambda,\cdot,\omega) &= -\tfrac{1}{2}\sqrt{\tfrac{n+2}{n}}(\tfrac{1}{2}+i\lambda)d^*\phi_q^{q+1}(\lambda,\cdot,\omega) \\ &= \tfrac{1}{2}(\lambda^2+\tfrac{1}{4})\phi_q^q(\lambda,\cdot,\omega). \end{aligned}$$

On the other hand, using (4.28) and (4.18),

$$\begin{aligned} dd^*\phi_q^{q,\pm}(\lambda,\cdot,\omega) &= \pm\tfrac{1}{2}i^{q^2}\sqrt{\tfrac{n+2}{n}}(\tfrac{1}{2}+i\lambda)d\phi_{q-1}^{q-1}(\lambda,\cdot,*\omega) \\ &= \pm\tfrac{1}{2}i^{q^2}\sqrt{\tfrac{n+2}{n}}(\lambda^2+\tfrac{1}{4})\phi_{q-1}^q(\lambda,\cdot,*\omega). \end{aligned}$$

Hence:

$$\Delta\phi_q^{q,\pm}(\lambda,\cdot,\omega) = \tfrac{1}{2}(\lambda^2+\tfrac{1}{4})\{\phi_q^q(\lambda,\cdot,\omega) \pm i^{q^2}\phi_{q-1}^q(\lambda,\cdot,*\omega)\}.$$

But (4.32) implies:

$$\begin{aligned} i^{q^2}\phi_{q-1}^q(\lambda,\cdot,*\omega) &= i^{q^2}(-1)^q*\{\phi_q^{q,+}(\lambda,\cdot,\omega)+\phi_q^{q,-}(\lambda,\cdot,\omega)\} \\ &= i^{q^2+2q}i^{q^2}\{\phi_q^{q,+}(\lambda,\cdot,\omega)-\phi_q^{q,-}(\lambda,\cdot,\omega)\} \\ &= \phi_q^{q,+}(\lambda,\cdot,\omega)-\phi_q^{q,-}(\lambda,\cdot,\omega). \end{aligned}$$

As a consequence,

$$\begin{aligned} &\Delta\phi_q^{q,\pm}(\lambda,\cdot,\omega) \\ &\quad= \tfrac{1}{2}(\lambda^2+\tfrac{1}{4})\{\phi_q^{q,+}(\lambda,\cdot,\omega)+\phi_q^{q,-}(\lambda,\cdot,\omega) \pm (\phi_q^{q,+}(\lambda,\cdot,\omega)-\phi_q^{q,-}(\lambda,\cdot,\omega))\} \\ &\quad= (\lambda^2+\tfrac{1}{4})\phi_q^{q,\pm}(\lambda,\cdot,\omega). \end{aligned}$$

The same argument works for $\phi_{q-1}^{q,\pm}$. ✓

Remark: as mentioned in the introduction, several authors have studied Poisson transforms in our context, and some results of this section can be partially found in previous works: Gaillard ([Gai86], Theorem 2'(c)) was the first to get the differential equations verified by the function $\phi_p^p(\lambda,\cdot)$ (see also [Gai88]); van der Ven ([Ven93], Section 9) described plainly the eigenspaces of the basic differential operators as images of the Poisson transforms $\Phi_q^p(\lambda,\cdot)$ — but only in the generic case $p < \frac{n-1}{2}$. Strichartz ([Str89], Section 7) obtained also that some Poisson transforms are eigenforms for the Laplacian, with the same eigenvalues as ours. Schuster [Sch87] showed that eigenforms for the Laplacian are eigenforms for some mean value operators. Finally (only in the case n odd), one can find in [BOS94], Corollary 4.4, differential equations for Fourier transforms of forms that are related to some equations in our Corollary 4.7.

5 Spherical functions associated with $\wedge^p H^n(\mathbb{R})$

5.1 Radial functions

It is well-known (see [Hel84, Hel94]) that the Fourier analysis of functions — i.e. of 0-forms — on Riemannian symmetric spaces of noncompact type G/K can be reduced to the spherical case, that is to the Fourier analysis of left K-invariant functions on G/K (or bi-K-invariant functions on G). One can proceed in the same way for differential p-forms on $H^n(\mathbb{R})$ — and more generally for sections of homogeneous vector bundles on G/K (see [Cam97a, Cam97b]) —, by considering the analogue of bi-K-invariant functions.

Let κ_1 and κ_2 be two irreducible unitary (finite dimensional) representations of K on the Hilbert spaces $\mathcal{H}_{\kappa_1}$ and $\mathcal{H}_{\kappa_2}$. We will say that a function $F : G \to \mathrm{Hom}(\mathcal{H}_{\kappa_1}, \mathcal{H}_{\kappa_2})$ is *of left type* κ_1 *and of right type* κ_2 — or more simply *of type* (κ_1, κ_2) — if

$$F(k_1 g k_2) = \kappa_2(k_2^{-1}) F(g) \kappa_1(k_1^{-1}) \qquad (\forall g \in G,\ \forall k_1, k_2 \in K).$$

When $\kappa_1 = \kappa_2 = \tau$, we will say that F is *τ-radial*. In our context, it is natural to consider τ_p-radial (or $\tau_p^{\pm}$-radial, when $p = n/2$) functions on G. Let us remark the following facts:

(i) if F is τ_p-radial, for all $\xi \in \mathcal{H}_{\tau_p}$, the map $g \mapsto F(g)\xi$ is of (right) type τ_p, and is thus identified to a p-form on G/K;

(ii) a τ_p-radial (resp. $\tau_{n/2}^{\pm}$-radial) function F is uniquely determined by any of the functions of type τ_p (resp. $\tau_{n/2}^{\pm}$) $g \mapsto F(g)\xi$, as long as $\xi \neq 0$;

(iii) thanks to the Cartan decomposition $G = KAK$, a τ_p-radial (resp. $\tau_{n/2}^{\pm}$-radial) function is determined by its restriction to the subgroup A (and even to $\overline{A^+} \simeq \mathbb{R}_+$) of G.

We shall denote by $\Gamma(G, K, \tau, \tau)$ the space of τ-radial functions on G and, as usual, shall replace 'Γ' by 'C', 'C_c', 'C^∞', 'C_c^∞' or 'L^2' when needed. The L^2 inner product on τ_p-radial functions is defined by:

$$\big(F, H\big) = \int_G dg\ \big(F(g), H(g)\big)_{HS} := \int_G dg\ \mathrm{tr}\{F(g)H(g)^*\}, \tag{5.1}$$

where $H(g)^* = {}^t\overline{H(g)}$ denotes the adjoint endomorphism of $H(g)$ and the subscript *'HS'* stands for 'Hilbert-Schmidt'.

In this paragraph, we want to study a particular class of τ-radial functions: the τ-spherical functions. As already observed in the previous sections, our results will depend on the (ir)reducibility of the representations τ_p and σ_p. This is why we shall divide, from now on, our study in three parts: we refer to the expression *special cases* for the cases where n is odd and $p = \frac{n-1}{2}$ or n even and $p = \frac{n}{2}$, and to the expression *generic case* for all other possibilities ($1 \leqslant p \leqslant \frac{n}{2} - 1$) — for simplicity, we will not deal with the well-known case $p = 0$.

Before getting to the heart of the matter, we recall a general result due essentially to Deitmar (see [Dei90], Theorem 3, Corollary 1 and Proposition 3).

Proposition 5.1. *Let G be a semisimple, connected, noncompact Lie group with finite centre, and K a maximal compact subgroup of G. For any irreducible unitary representation τ of K, let us equip the space $C_c^{(\infty)}(G, K, \tau, \tau)$ with the following convolution product:*

$$(F * H)(x) = \int_G dy\, F(y^{-1}x)H(y) = \int_G dy\, F(y)H(xy^{-1}). \tag{5.2}$$

Then the following assertions are equivalent:

(a) *the K-type τ appears with multiplicity $\leqslant 1$ in any irreducible unitary (and even admissible) representation of G;*
(b) *$\tau_{|M}$ is multiplicity free;*
(c) *equipped with the convolution product (5.2), $C_c^{(\infty)}(G, K, \tau, \tau)$ is a commutative algebra;*
(d) *$\mathbb{D}(G, K, \tau)$ is a commutative algebra.*

Moreover, the algebra $U(\mathfrak{g})^K$ is commutative if and only if $\mathfrak{g} = \mathfrak{o}(n, 1)$ or $\mathfrak{su}(n, 1)$, and in this case the previous conditions are satisfied for every $\tau \in \widehat{K}$.

This result implies immediately the commutativity of $C_c^{(\infty)}(G, K, \tau, \tau)$ for $\tau = \tau_p$ ($p < n/2$) and $\tau = \tau^{\pm}_{\frac{n}{2}}$. It gives also another proof of the commutativity of $\mathbb{D}(G, K, \tau)$, for the same representations τ (see Corollary 2.2).

Remark: let us try to shed some light on the confusion prevailing in the literature about the contents of Proposition 5.1.

1. The link existing between the commutativity of the algebra $C_c(G,K,\tau,\tau)$ and the multiplicity of τ in any irreducible unitary representation of G goes back to Godement [God52]. More exactly, assuming only that G is a unimodular locally compact group, Godement showed the equivalence (a) $\Leftrightarrow$ (c) for the algebra $L^o(\tau)$ of K-central functions $f \in C_c(G)$ verifying $\overline{\chi_\tau} * f = f * \overline{\chi_\tau} = f$, where $\chi_\tau(k) = \dim(\tau)\,\mathrm{tr}\,\tau(k)$. On the other hand, $L^o(\tau)$ and $C_c(G,K,\tau,\tau)$ are isomorphic (see [War72], vol. II, Section 6).

2. A quick and elegant proof of (a) for $G = SO_e(n,1)$ or $SU(n,1)$ (as well as $U(n,1)$, $SO(n+1)$, *etc.*) has been given by Koornwinder [Koo82], by showing that $(G \times K, \mathrm{diag}\,K)$ is a Gelfand pair.

3. Cooper [Coo75] was the first to show that $U(\mathfrak{g})^K$ can be identified to the product of the centres $Z(\mathfrak{g}) \cdot Z(\mathfrak{k})$ (and is, for that reason, commutative) when $\mathfrak{g} = \mathfrak{o}(n,1)$, $\mathfrak{su}(n,1)$ or $\mathfrak{u}(n,1)$. Another proof of this result has been obtained by Benabdallah [Ben83] and Sezionale Basilicato [SB83] by exhibiting explicitly a system of generators of $U(\mathfrak{g})^K$. A third kind of proof (shorter) has been proposed by Johnson [Joh89] and Knop [Kno90]. Finally, one may also consult [Tir92], where other results on the structure of $U(\mathfrak{g})^K$ are stated.

Generic case

We have seen above that τ_p-radial functions F are determined by their restriction $F_{|A}$ to the subgroup A of G. But, since A and M commute, for all $m \in M$ and all $a \in A$,

$$\tau_p(m)F(a) = F(am^{-1}) = F(m^{-1}a) = F(a)\tau_p(m),$$

which shows that $F_{|A}$ is a M-endomorphism of $\wedge^p\mathbb{C}^n$. Since $\tau_{p|M}$ is multiplicity free in the generic case, Schur's lemma implies that $F_{|A}$ is scalar on each M-irreducible component of $\wedge^p\mathbb{C}^n$, and we know that such nontrivial subspaces are either $\mathcal{H}_{\sigma_p} \simeq \wedge^p\mathbb{C}^{n-1}$ or $\mathcal{H}_{\sigma_{p-1}} \simeq e_1 \wedge \wedge^{p-1}\mathbb{C}^{n-1}$, the representations σ_{p-1} and σ_p being inequivalent. Thus,

$$F(a_t)\, e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_p} = \begin{cases} f_{p-1}(t)\, e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_p} & \text{if } i_1 = 1, \\ f_p(t)\, e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_p} & \text{if } i_1 > 1, \end{cases}$$

where the functions $f_p, f_{p-1} : \mathbb{R} \to \mathbb{C}$ defined by this expression are even, for the nontrivial element $w = m'M$ of the Weyl group $W(\mathfrak{g},\mathfrak{a})$ verifies $m'^{-1}a_t m' = a_{-t}$ and stabilizes the M-invariant subspaces $\mathcal{H}_{\sigma_p}$ and $\mathcal{H}_{\sigma_{p-1}}$ (see §3.2). The functions f_{p-1} and f_p will be called the *scalar components of* F, and we will remember that they determine completely F.

As announced in Section 2, p. 47, we now change the normalization of the Haar measure on G, in order to adopt the notations of [Koo84], which we will use continually in the sequel.
If dg denotes the Haar measure on G introduced in Section 2, we put

$$dx = 2^{n-2}\pi^{-\frac{n}{2}}\Gamma(\tfrac{n}{2})dg$$

— the same notation dx will be kept for the measure on $H^n(\mathbb{R})$ if there is no ambiguity —, so that we have the following integral formula for the Cartan decomposition of G:

$$\int_G dx\, f(x) = \int_K dk_1 \int_0^\infty dt\,(2\operatorname{sh} t)^{n-1} \int_K dk_2\, f(k_1 a_t k_2). \tag{5.3}$$

(From now on, to be coherent, x will be the variable of the group G.) We can reduce then the definition (5.1) to an integral over $\overline{A^+}$ (that is, over $[0,\infty[$) by using formula (5.3): since

$$\operatorname{tr}\{F(k_1 a_t k_2)H(k_1 a_t k_2)^*\} = \operatorname{tr}\{F(a_t)H(a_t)^*\},$$

and since the operators $F(a_t)$ and $H(a_t)$ are scalar on the two M-invariant subspaces of $\wedge^p\mathbb{C}^n$, one gets an expression depending only on the scalar components:

$$\big(F, H\big) = \int_0^\infty dt\,(2\operatorname{sh} t)^{n-1}\{C_{n-1}^{p-1} f_{p-1}(t)\overline{h_{p-1}}(t) + C_{n-1}^{p} f_p(t)\overline{h_p}(t)\}. \tag{5.4}$$

Special case $p = \frac{n-1}{2}$

The difference with the generic case is that σ_p is no more irreducible: $\sigma_p = \sigma_p^+ \oplus \sigma_p^-$, and the restriction of τ_p to M splits into three inequivalent components: $\tau_{p|_M} = \sigma_{p-1} \oplus \sigma_p^+ \oplus \sigma_p^-$. It follows that the restriction $F_{|A}$ of a τ_p-radial function F to A is scalar on each subspace $e_1 \wedge \wedge^{p-1}\mathbb{C}^{n-1}$, $\wedge_+^p\mathbb{C}^{n-1}$ and $\wedge_-^p\mathbb{C}^{n-1}$. Proceeding as in the generic case, one can see that to F correspond now three complex-valued functions f_{p-1}, f_p^+, f_p^- on $\mathbb{R}$, which verify the conditions:

$$f_{p-1} \text{ is even}, \quad f_p^+(t) = f_p^-(-t) \quad (\forall t \in \mathbb{R}).$$

Indeed, if $w = m'M$ is the nontrivial element of the Weyl group $W(\mathfrak{g},\mathfrak{a})$, the operator $\tau_p(m')$ fixes the subspace $\mathcal{H}_{\sigma_{p-1}}$ of $\mathcal{H}_{\tau_p}$ (as in the generic case), but exchanges the subspaces $\mathcal{H}_{\sigma_p^+}$ and $\mathcal{H}_{\sigma_p^-}$ (see §3.2). Thus, for $\xi^+ \in \mathcal{H}_{\sigma_p^+}$, on the one hand $F(a_{-t})\xi^+ = f_p^+(-t)\xi^+$, and on the other hand:

$$F(a_{-t})\xi^+ = F(m'^{-1}a_t m')\xi^+ = \tau_p(m')^{-1}F(a_t)\tau_p(m')\xi^+ = f_p^-(t)\xi^+.$$

(The calculation with an element $\xi^- \in \mathcal{H}_{\sigma_p^-}$ gives the same result.)

We define the convolution product and the scalar product as in (5.2) and (5.1). Note that (5.1) becomes:

$$\begin{aligned}\big(F,H\big) = \int_0^\infty dt\,(2\operatorname{sh} t)^{n-1}\{C_{n-1}^{p-1} f_{p-1}(t)\overline{h_{p-1}}(t) + \tfrac{1}{2}C_{n-1}^{p} f_p^+(t)\overline{h_p^+}(t) \\ + \tfrac{1}{2}C_{n-1}^{p} f_p^-(t)\overline{h_p^-}(t)\}.\end{aligned} \tag{5.5}$$

Special case $p = \frac{n}{2}$

When $p = \frac{n}{2}$, τ_p is no more irreducible. More precisely, we recall the decompositions of $\tau_{\frac{n}{2}}$:

- K-decomposition of $\wedge^{\frac{n}{2}}\mathbb{C}^n$

$$\begin{cases} \wedge^{\frac{n}{2}}\mathbb{C}^n = \wedge_+^{\frac{n}{2}}\mathbb{C}^n \oplus \wedge_-^{\frac{n}{2}}\mathbb{C}^n, \\ \tau_{\frac{n}{2}|K} = \tau_{\frac{n}{2}}^+ \oplus \tau_{\frac{n}{2}}^-; \end{cases}$$

(decomposition into inequivalent irreducible factors.)

- M-decomposition of $\wedge^{\frac{n}{2}}\mathbb{C}^n$

$$\begin{cases} \wedge^{\frac{n}{2}}\mathbb{C}^n = e_1 \wedge \wedge^{\frac{n}{2}-1}\mathbb{C}^{n-1} \,\widetilde{\oplus}\, \wedge^{\frac{n}{2}}\mathbb{C}^{n-1} = \wedge_+^{\frac{n}{2}}\mathbb{C}^n \,\widetilde{\oplus}\, \wedge_-^{\frac{n}{2}}\mathbb{C}^n, \\ \tau_{\frac{n}{2}|M} = \sigma_{\frac{n}{2}-1} \,\widetilde{\oplus}\, \sigma_{\frac{n}{2}} = \tau_{\frac{n}{2}|M}^+ \,\widetilde{\oplus}\, \tau_{\frac{n}{2}|M}^-. \end{cases}$$

(decomposition into irreducible factors, all equivalent.)

If we consider $\tau_{\frac{n}{2}}$-radial functions F, we obtain M-endomorphisms of $\wedge^{\frac{n}{2}}\mathbb{C}^n$, when restricting to A. Thus, F has four non canonically defined scalar components, which is awkward. Therefore, we will consider instead $\tau_{\frac{n}{2}}^\pm$-radial functions $F^\pm$ which, by virtue of the M-decomposition above, will have only one scalar component $f^\pm$ on $\wedge_\pm^{\frac{n}{2}}\mathbb{C}^n$.

The scalar product on $L^2(G,K,\tau_{\frac{n}{2}}^\pm,\tau_{\frac{n}{2}}^\pm)$ is then simply given by

$$\big(F^\pm,H^\pm\big) = \tfrac{1}{2}\,C_n^{n/2}\int_0^\infty dt\,(2\operatorname{sh} t)^{n-1} f^\pm(t)\overline{h^\pm}(t).$$

5.2 Construction and description of spherical functions

We are now interested in a special class of τ-radial functions on G, the so-called *τ-spherical functions*. It is well-known that spherical functions play an essential role in the development of general harmonic analysis theory. For example, the spherical analysis on Riemannian noncompact symmetric spaces is based on the abstract spherical function theory on *Gelfand pairs*. Recall that (G, K) is called a Gelfand pair if

(i) G is a locally compact group, K a compact subgroup of G, and

(ii) the convolution algebra $C_c(G)^\natural = C_c(G, K, 1, 1)$ of continuous bi-K-invariant functions on G with compact support is commutative.

In our Appendix B, we extend this notion as follows:

(i) G is a unimodular locally compact group, K a compact subgroup of G, and

(ii)' τ is an irreducible unitary representation of K such that the convolution algebra $C_c(G, K, \tau, \tau)$ is commutative.

We say then that (G, K, τ) is a *Gelfand triple*, and we develop in this setting a generalization of the spherical function theory on Gelfand pairs.

Now, let $\tau \in \{\tau_1, \tau_2, \ldots, \tau_{\frac{n-1}{2}}, \tau_{\frac{n}{2}}^{\pm}\}$. Since $(SO_e(n,1), SO(n), \tau)$ is a Gelfand triple by Proposition 5.1, we shall use in this section some results of Appendix B. We incite then the reader to have a look at this appendix before proceeding further — we point out in particular Theorem B.2, Lemma B.7, Proposition B.9, Theorem B.12, Proposition B.14 and Proposition B.15.

Generic case

With the notations of Section 4, we put, for $\lambda \in \mathbb{C}$:

$$\Phi_q^p(\lambda, \cdot) : G \longrightarrow \mathrm{End}(\wedge^p \mathbb{C}^n),$$
$$\Phi_q^p(\lambda, x) = P_q^p \circ \pi_{\sigma_q, \lambda}(x^{-1}) \circ J_q^p, \tag{5.6}$$

for $p - q = 0, 1$ [1]. Then $\Phi_q^p(\lambda, \cdot)$ is τ_p-spherical by Proposition B.9. We know also by Theorem B.12 that it is an eigenfunction for the algebra $\mathbb{D}(G, K, \tau_p)$. But we can

[1] In Section 4 we defined two differential forms ϕ_p^p and ϕ_p^{p+1} on G/K with respective degrees p and $p+1$ starting from a p-form on the boundary K/M. Here, our point of view is rather to obtain, for a fixed integer p, the two τ_p-spherical functions constructed with forms on the boundary of degree p or $p-1$. This is why we will exchange, from now on, the roles of the indices p and q.

have more precise information by remarking that $\Phi_q^p(\lambda,x)\xi = \phi_q^p(\lambda,x,\omega)$ if ω denotes the q-form on K/M defined by $\omega = J_q^p(\xi,\cdot)$. Thus Φ_q^p verifies certain differential equations of Corollary 4.7, and in particular:

$$\Phi = \Phi_p^p(\lambda,\cdot) \text{ verifies } \begin{cases} \Delta\Phi = \{\lambda^2 + (\rho - p)^2\}\Phi, & (5.7)\\ d^*\Phi = 0, & (5.8)\end{cases}$$

$$\text{and } \Phi = \Phi_{p-1}^p(\lambda,\cdot) \text{ verifies } \begin{cases} \Delta\Phi = \{\lambda^2 + [\rho - (p-1)]^2\}\Phi, & (5.9)\\ d\Phi = 0. & (5.10)\end{cases}$$

Remarks:

1. When there is no ambiguity, we may drop the variable λ in the expression Φ_q^p.

2. As matrix blocks of the representation $\pi_{\sigma_q,\lambda}$, the functions Φ_q^p arise naturally as generalizations of 'scalar' spherical functions ($p=0$), which are matrix coefficients.

3. The unitarity of $\pi_{\sigma_q,\lambda}$ for $\lambda\in\mathbb{R}$ gives the identity $\pi_{\sigma_q,\lambda}(x)^* = \pi_{\sigma_q,\lambda}(x^{-1})$ for all $\lambda\in\mathbb{R}$ and, by analytic continuation, $\pi_{\sigma_q,\lambda}(x)^* = \pi_{\sigma_q,\bar\lambda}(x^{-1})$ for all $\lambda\in\mathbb{C}$. As a consequence, $\Phi_q^p(\lambda,x)^* = \Phi_q^p(\bar\lambda,x^{-1})$ for all $\lambda\in\mathbb{C}$.

4. On the other hand, the action of the Weyl group W on the principal series $\pi_{\sigma_q,\lambda}$ implies that the functions $\lambda\mapsto\Phi_q^p(\lambda,x)$ are even. Indeed, when λ is real and if w is the nontrivial Weyl group element, $\pi_{w\sigma_q,w\lambda} = \pi_{\sigma_q,\lambda}$ (see [Kna86], §VII.4). Thus $\Phi_q^p(-\lambda,\cdot) = \Phi_q^p(w\lambda,\cdot) = \Phi_q^p(\lambda,\cdot)$ for real λ first, and then for all complex λ by analytic continuation.

Recall that $\Phi_q^p(\lambda,\cdot)$ admits the following representation as Eisenstein integral (see Proposition B.14):

$$\Phi_q^p(\lambda,x) = \frac{\dim\tau_p}{\dim\sigma_q}\int_K dk\, e^{-(i\lambda+\rho)H(xk)}\tau_p(k)\circ P_{\sigma_q}\circ\tau_p(\underline{k}(xk))^{-1}.$$

Our goal in this section is to examine the effect of differential equations (5.7) to (5.10) on the scalar components of the functions $\Phi_q^p(\lambda,\cdot)$ and to point out that — as in the case $p=0$, see [Koo84] — they can be expressed in a simple way in terms of Jacobi functions.

Before stating next result, notice that equations (5.7) and (5.9) are formally equivalent; for instance, one gets the second one from the first one by replacing λ^2 by $\lambda^2 - n + 2p$.

Lemma 5.2. *Let p be generic. The differential equation (5.7) verified by $\Phi = \Phi_p^p(\lambda,\cdot)$ corresponds to the following equations for its scalar components $\varphi_{p-1}(\lambda,\cdot)$, $\varphi_p(\lambda,\cdot)$:*

(i) *on the subspace $e_1 \wedge \wedge^{p-1}\mathbb{C}^{n-1}$ of $\wedge^p\mathbb{C}^n$,*

$$\begin{aligned}\frac{d^2}{dt^2}\varphi_{p-1}(\lambda,t) + (n-1)(\coth t)\frac{d}{dt}\varphi_{p-1}(\lambda,t) + (\rho^2+\lambda^2-n+2p)\varphi_{p-1}(\lambda,t)\\ - 2(n-p)(\operatorname{sh} t)^{-2}\varphi_{p-1}(\lambda,t) + 2(n-p)(\operatorname{sh} t)^{-1}(\coth t)\varphi_p(\lambda,t) = 0; \quad (5.11)\end{aligned}$$

(ii) *on the subspace $\wedge^p\mathbb{C}^{n-1}$ of $\wedge^p\mathbb{C}^n$,*

$$\begin{aligned}\frac{d^2}{dt^2}\varphi_{p}(\lambda,t) + (n-1)(\coth t)\frac{d}{dt}\varphi_{p}(\lambda,t) + (\rho^2+\lambda^2)\varphi_{p}(\lambda,t)\\ - 2p(\operatorname{sh} t)^{-2}\varphi_{p}(\lambda,t) + 2p(\operatorname{sh} t)^{-1}(\coth t)\varphi_{p-1}(\lambda,t) = 0. \quad (5.12)\end{aligned}$$

In the same way, the differential equation (5.9) corresponds, for the scalar components $\varphi_{p-1}(\mu,\cdot)$, $\varphi_p(\mu,\cdot)$ of $\Phi = \Phi_{p-1}^p(\lambda,\cdot)$, to two differential equations of order 2, obtained from the previous ones by replacing λ^2 by λ^2+n-2p.

Proof: in Section 4, we introduced the subspaces $\bar{\mathfrak{n}} = \theta(\mathfrak{n})$, $\mathfrak{q}$, $\mathfrak{l}$ of $\mathfrak{g}$ and we reminded the identity $\mathfrak{n}\oplus\bar{\mathfrak{n}} = \mathfrak{q}\oplus\mathfrak{l}$. Let us denote by $\widetilde{X}_j$, $\theta\widetilde{X}_j$, $\widetilde{Y}_j$ and $\widetilde{Z}_j$ ($j=2,\dots,n$) the respective basis elements in $\mathfrak{n}$, $\bar{\mathfrak{n}}=\theta(\mathfrak{n})$, $\mathfrak{q}$ and $\mathfrak{l}$:

$$\widetilde{X}_j = \begin{pmatrix} 0 & \dots & 1 & \dots & 0\\ \vdots & & \vdots & & \vdots\\ -1 & \dots & 0 & \dots & 1\\ \vdots & & \vdots & & \vdots\\ 0 & \dots & 1 & \dots & 0\end{pmatrix}, \qquad \theta\widetilde{X}_j = \begin{pmatrix} 0 & \dots & 1 & \dots & 0\\ \vdots & & \vdots & & \vdots\\ -1 & \dots & 0 & \dots & -1\\ \vdots & & \vdots & & \vdots\\ 0 & \dots & -1 & \dots & 0\end{pmatrix},$$

$$\widetilde{Y}_j = \frac{\widetilde{X}_j - \theta\widetilde{X}_j}{2} = \begin{pmatrix} 0 & \dots & 0 & \dots & 0\\ \vdots & & \vdots & & \vdots\\ 0 & \dots & 0 & \dots & 1\\ \vdots & & \vdots & & \vdots\\ 0 & \dots & 1 & \dots & 0\end{pmatrix} = e_j, \qquad \widetilde{Z}_j = \frac{\widetilde{X}_j + \theta\widetilde{X}_j}{2} = \begin{pmatrix} 0 & \dots & 1 & \dots & 0\\ \vdots & & \vdots & & \vdots\\ -1 & \dots & 0 & \dots & 0\\ \vdots & & \vdots & & \vdots\\ 0 & \dots & 0 & \dots & 0\end{pmatrix},$$

the nonzero coefficients figuring in the j-th lines and columns.
Using Kuga's formula (4.15) and the formula giving the action of $\Omega_{\mathfrak{k}}$ on τ_p-spherical functions $\Phi = \Phi_p^p$ (4.17), we get:

$$\Delta\Phi(x) = -\Phi(x:\Omega_{\mathfrak{p}}) - p(n-p)\Phi(x).$$

We have then to compute explicitly the value of

$$\Phi(x : \Omega_{\mathfrak{p}}) = \Phi(x : C_0^2) + \sum_{j=2}^{n} \Phi(x : \widetilde{Y}_j^2),$$

where C_0 was defined in (2.1). As we can restrict to $x = a_t \in A$, and as

$$\begin{aligned}(\operatorname{Ad} a_{-t})\widetilde{Z}_j &= \tfrac{1}{2}e^{-t}\widetilde{X}_j + \tfrac{1}{2}e^{t}\theta\widetilde{X}_j \\ &= (\operatorname{ch} t)\widetilde{Z}_j - (\operatorname{sh} t)\widetilde{Y}_j,\end{aligned}$$

one deduces the expression of $\widetilde{Y}_j$ in $(\operatorname{Ad} a_{-t})\mathfrak{l} \oplus \mathfrak{l}$:

$$\widetilde{Y}_j = -(\operatorname{sh} t)^{-1}(\operatorname{Ad} a_{-t} \cdot \widetilde{Z}_j) + (\coth t)\widetilde{Z}_j.$$

Then, in the enveloping algebra,

$$\begin{aligned}\widetilde{Y}_j^2 = (\operatorname{sh} t)^{-2}(\operatorname{Ad} a_{-t} \cdot \widetilde{Z}_j)^2 + (\coth t)^2 \widetilde{Z}_j^2 \\ - 2(\operatorname{sh} t)^{-1}(\coth t)(\operatorname{Ad} a_{-t} \cdot \widetilde{Z}_j)\widetilde{Z}_j + (\coth t)C_0,\end{aligned}$$

since $[(\operatorname{Ad} a_{-t} \cdot \widetilde{Z}_j), \widetilde{Z}_j] = -(\operatorname{sh} t)[\widetilde{Y}_j, \widetilde{Z}_j] = (\operatorname{sh} t)C_0$. Therefore,

$$\begin{aligned}\Delta\Phi(a_t) &= -\Phi(a_t : C_0^2) - \sum_{j=2}^{n} \Phi(a_t : \widetilde{Y}_j^2) - p(n-p)\Phi(a_t) \\ &= -\frac{d^2}{dt^2}\Phi(a_t) - (n-1)(\coth t)\frac{d}{dt}\Phi(a_t) - (\operatorname{sh} t)^{-2}\Phi(a_t)\sum_{j=2}^{n} \tau_p(\widetilde{Z}_j)^2 \\ &\quad - (\coth t)^2 \sum_{j=2}^{n} \tau_p(\widetilde{Z}_j)^2 \Phi(a_t) \\ &\quad + 2(\operatorname{sh} t)^{-1}(\coth t)\sum_{j=2}^{n} \tau_p(\widetilde{Z}_j)\Phi(a_t)\tau_p(\widetilde{Z}_j) - p(n-p)\Phi(a_t). \qquad (5.13)\end{aligned}$$

The first step consists in describing the action of $\tau_1(\widetilde{Z}_j)$ on the basis elements e_i:

$$\tau_1(\widetilde{Z}_j) : \begin{cases} e_1 \longmapsto -e_j, \\ e_j \longmapsto e_1, \\ e_i \longmapsto 0 \qquad \text{if } i \neq 1, j. \end{cases}$$

Let us distinguish two cases:

- if $\xi = e_1 \wedge e_{i_2} \wedge \ldots \wedge e_{i_p} \in e_1 \wedge \wedge^{p-1}\mathbb{C}^{n-1}$:

$$\tau_p(\widetilde{Z}_j)\xi = \begin{cases} -e_j \wedge e_{i_2} \wedge \ldots \wedge e_{i_p} & \text{if } j \notin \{i_2, \ldots, i_p\}, \\ 0 & \text{if } j = i_r \text{ for some } r \geqslant 2. \end{cases}$$

There are then $(n-1)-(p-1) = n-p$ 'non-nullity' cases for $\tau_p(\widetilde{Z}_j)\xi$, and

$$\begin{aligned} \tau_p(\widetilde{Z}_j)\Phi(a_t)\tau_p(\widetilde{Z}_j)\xi &= -\tau_p(\widetilde{Z}_j)\varphi_p(t) e_j \wedge e_{i_2} \wedge \ldots \wedge e_{i_p} \\ &= -\varphi_p(t)\xi. \end{aligned}$$

- if $\xi = e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_p} \in \wedge^p\mathbb{C}^{n-1}$ (with $i_1 \geqslant 2$):

$$\tau_p(\widetilde{Z}_j)\xi = \begin{cases} e_1 \wedge i_{e_j}(\xi) & \text{if } j = i_r \text{ for some } r \geqslant 1, \\ 0 & \text{else.} \end{cases}$$

There are here p 'non-nullity' cases for $\tau_p(\widetilde{Z}_j)\xi$, and

$$\begin{aligned} \tau_p(\widetilde{Z}_j)\Phi(a_t)\tau_p(\widetilde{Z}_j)\xi &= (-1)^{r-1}\tau_p(\widetilde{Z}_j)\varphi_{p-1}(t) e_1 \wedge e_{i_1} \wedge \ldots \wedge \widehat{e}_{i_r} \wedge \ldots \wedge e_{i_p} \\ &= (-1)^r \varphi_{p-1}(t) e_j \wedge e_{i_1} \wedge \ldots \wedge \widehat{e}_{i_r} \wedge \ldots \wedge e_{i_p} \\ &= -\varphi_{p-1}(t)\xi. \end{aligned}$$

To sum up,

$$\sum_{j=2}^{n} \tau_p(\widetilde{Z}_j)\Phi(a_t)\tau_p(\widetilde{Z}_j) = \begin{cases} -(n-p)\varphi_p(t)\,\mathrm{Id} & \text{on } e_1 \wedge \wedge^{p-1}\mathbb{C}^{n-1}, \\ -p\,\varphi_{p-1}(t)\,\mathrm{Id} & \text{on } \wedge^p\mathbb{C}^{n-1}. \end{cases} \tag{5.14}$$

One can make $\tau_p(\widetilde{Z}_j)^2$ explicit in a similar way or, more simply, use the relation

$$\sum_{j=2}^{n} \tau_p(\widetilde{Z}_j)^2 = \tau_p(\Omega_{\mathfrak{l}}) = \tau_p(\Omega_{\mathfrak{k}}) - \tau_p(\Omega_{\mathfrak{m}}),$$

with

$$\begin{aligned} &\tau_p(\Omega_{\mathfrak{k}}) = -p(n-p)\,\mathrm{Id}, \\ \text{and } &\tau_p(\Omega_{\mathfrak{m}}) = \begin{cases} \sigma_{p-1}(\Omega_{\mathfrak{m}}) = -(p-1)(n-p)\,\mathrm{Id} & \text{on } e_1 \wedge \wedge^{p-1}\mathbb{C}^{n-1}, \\ \sigma_p(\Omega_{\mathfrak{m}}) = -p(n-1-p)\,\mathrm{Id} & \text{on } \wedge^p\mathbb{C}^{n-1}, \end{cases} \end{aligned}$$

since $\sigma_q(\Omega_{\mathfrak{l}})$ is the analogue in $\wedge^q\mathbb{C}^{n-1}$ of the operator $\tau_p(\Omega_{\mathfrak{k}})$ in $\wedge^p\mathbb{C}^n$. As a consequence,

$$\tau_p(\Omega_{\mathfrak{l}}) = \begin{cases} -(n-p)\,\mathrm{Id} & \text{on } e_1 \wedge \wedge^{p-1}\mathbb{C}^{n-1}, \\ -p\ \mathrm{Id} & \text{on } \wedge^p\mathbb{C}^{n-1}. \end{cases} \tag{5.15}$$

Putting (5.14) and (5.15) in (5.13), one obtains the announced expressions (5.11) and (5.12) of (5.7) on $e_1 \wedge \wedge^{p-1}\mathbb{C}^{n-1}$ and $\wedge^p\mathbb{C}^{n-1}$. ✓

Lemma 5.3. *Let p be generic. The equation (5.10) corresponds, for the scalar components* $\varphi_{p-1}(\lambda,\cdot)$, $\varphi_p(\lambda,\cdot)$ *of* $\Phi = \Phi^p_{p-1}(\lambda,\cdot)$, *to the equation*

$$\frac{d}{dt}\varphi_p(\lambda,t) = -p(\coth t)\varphi_p(\lambda,t) + p(\operatorname{sh} t)^{-1}\varphi_{p-1}(\lambda,t). \tag{5.16}$$

Similarly, the equation (5.8) is equivalent to the following equation for scalar components $\varphi_{p-1}(\lambda,\cdot)$, $\varphi_p(\lambda,\cdot)$ *of* $\Phi = \Phi^p_p(\lambda,\cdot)$*:*

$$\frac{d}{dt}\varphi_{p-1}(\lambda,t) = -(n-p)(\coth t)\varphi_{p-1}(\lambda,t) + (n-p)(\operatorname{sh} t)^{-1}\varphi_p(\lambda,t). \tag{5.17}$$

Proof : (5.10) means $\sum_{j=1}^n e_j \wedge \Phi(x : e_j)\xi = 0$ for all $x \in G$ and all $\xi \in \wedge^p\mathbb{C}^n$. Restricting as always to $x = a_t \in A$,

$$\begin{aligned} d\Phi(a_t)\xi &= e_1 \wedge \Phi(a_t : C_0)\xi + \sum_{j=2}^n e_j \wedge \Phi(a_t : \widetilde{Y}_j)\xi \\ &= e_1 \wedge \frac{d}{dt}\Phi(a_t)\xi - (\operatorname{sh} t)^{-1}\sum_{j=2}^n e_j \wedge \Phi(\widetilde{Z}_j : a_t)\xi + (\coth t)\sum_{j=2}^n e_j \wedge \Phi(a_t : \widetilde{Z}_j)\xi \\ &= e_1 \wedge \frac{d}{dt}\Phi(a_t)\xi + (\operatorname{sh} t)^{-1}\sum_{j=2}^n e_j \wedge \Phi(a_t)\tau_p(\widetilde{Z}_j)\xi \\ &\quad - (\coth t)\sum_{j=2}^n e_j \wedge \tau_p(\widetilde{Z}_j)\Phi(a_t)\xi. \end{aligned} \tag{5.18}$$

The expression of (5.18) on $\xi = e_1 \wedge \eta \in e_1 \wedge \wedge^{p-1}\mathbb{C}^{n-1}$ is:

$$\begin{aligned} d\Phi(a_t)\xi = {} & \frac{d}{dt}\varphi_{p-1}(t) e_1 \wedge e_1 \wedge \eta \\ & - (\operatorname{sh} t)^{-1}\varphi_p(t)\sum_{j=2}^n e_j \wedge e_j \wedge \eta + (\operatorname{sh} t)^{-1}\varphi_{p-1}(t)\sum_{j=2}^n e_j \wedge e_1 \wedge e_1 \wedge i_{e_j}(\eta) \\ & + (\coth t)\varphi_p(t)\sum_{j=2}^n e_j \wedge e_j \wedge \eta - (\coth t)\varphi_{p-1}(t)\sum_{j=2}^n e_j \wedge e_1 \wedge e_1 \wedge i_{e_j}(\eta). \end{aligned}$$

All the exterior products vanish, and:

$$d\Phi(a_t)\xi = 0. \tag{5.19}$$

Now, the expression of (5.18) on a vector $\xi \in \wedge^p \mathbb{C}^{n-1}$ gives:

$$d\Phi(a_t)\xi = \frac{d}{dt}\varphi_p(t)e_1 \wedge \xi + (\operatorname{sh} t)^{-1}\varphi_{p-1}(t)\sum_{j=2}^{n} e_j \wedge e_1 \wedge i_{e_j}(\xi)$$
$$- (\coth t)\varphi_p(t)\sum_{j=2}^{n} e_j \wedge e_1 \wedge i_{e_j}(\xi).$$

But $e_j \wedge e_1 \wedge i_{e_j}(\xi) = -e_1 \wedge \xi$. Hence:

$$d\Phi(a_t)\xi = \{\frac{d}{dt}\varphi_p(t) + p(\coth t)\varphi_p(t) - p(\operatorname{sh} t)^{-1}\varphi_{p-1}(t)\}e_1 \wedge \xi. \tag{5.20}$$

When $d\Phi = 0$, (5.19) is useless, but (5.20) implies (5.16).
To prove the second formula (5.17), we use duality arguments. Define the τ_{n-p}-radial function $\widetilde{\Phi}$ by:

$$\widetilde{\Phi}(x) = (-1)^{p(n-p)} {*}\Phi(x){*}.$$

Then, if the scalar components of Φ are φ_{p-1} and φ_p, the scalar components of $\widetilde{\Phi}$ are φ_p and φ_{p-1}. Moreover, an easy calculation shows that $d\widetilde{\Phi}(x) = (-1)^{np} {*} d^*\Phi(x){*}$. Thus $d^*\Phi = 0$ if and only if $d\widetilde{\Phi} = 0$. We can then use results (5.19) and (5.20), and exchange φ_p with φ_{p-1} and p with $n-p$. ✓

As claimed before, we shall see now that the scalar components of τ_p-spherical functions are (linear combinations of) Jacobi functions. We recall some facts about these special functions (for complements, see [Koo84], §2.1).

Let $\alpha \in \mathbb{C} \setminus \{-1, -2, \ldots\}$, $\beta \in \mathbb{C}$ and $\lambda \in \mathbb{C}$. Put

$$\gamma = \gamma(\alpha, \beta) = \alpha + \beta + 1. \tag{5.21}$$

The *Jacobi function* $\phi_\lambda^{(\alpha,\beta)}$ is, by definition, the hypergeometric function

$$\phi_\lambda^{(\alpha,\beta)}(t) = {}_2F_1(\tfrac{\gamma+i\lambda}{2}, \tfrac{\gamma-i\lambda}{2}; \alpha+1; -\operatorname{sh}^2 t) \qquad (t \in \mathbb{R}). \tag{5.22}$$

Consider the *weight*

$$\zeta_{\alpha,\beta}(t) = (2\operatorname{sh} t)^{2\alpha+1}(2\operatorname{ch} t)^{2\beta+1}, \tag{5.23}$$

as well as the differential operator of the second order ('Jacobi Laplacian')

$$L_{\alpha,\beta} = \frac{d^2}{dt^2} + \frac{\zeta'_{\alpha,\beta}(t)}{\zeta_{\alpha,\beta}(t)}\frac{d}{dt}$$
$$= \frac{d^2}{dt^2} + \{(2\alpha+1)\coth t + (2\beta+1)\operatorname{th} t\}\frac{d}{dt}. \tag{5.24}$$

Then $\phi = \phi^{(\alpha,\beta)}_{\pm\lambda}$ is the unique analytic solution on $\mathbb{R}$ of the equation

$$(L_{\alpha,\beta} + \lambda^2 + \gamma^2)\phi = 0 \tag{5.25}$$

which is even and verifies $\phi(0) = 1$.

Remark: the (scalar) spherical functions φ_λ on hyperbolic spaces $H^n(\mathbb{F})$ (for $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$) and $H^2(\mathbb{O})$ are closely related to the Jacobi functions $\phi^{(\alpha,\beta)}_\lambda$. Indeed, one has

$$\varphi_\lambda(a_t) = \phi_\lambda^{(\frac{dn}{2}-1,\frac{d}{2}-1)},$$

where $d = \dim_\mathbb{R} \mathbb{F}$.

Now, in the definition of a Jacobi function $\phi^{(\alpha,\beta)}_\lambda$, we add two more parameters $l, m \in \mathbb{Z}$, and put (see [Koo84], formula (4.15)):

$$\phi^{(\alpha,\beta)}_{\lambda,l,m}(t) = (\operatorname{sh} t)^l (\operatorname{ch} t)^m \phi^{(\alpha+l,\beta+m)}_\lambda(t). \tag{5.26}$$

One easily checks that $\phi = \phi^{(\alpha,\beta)}_{\lambda,l,m}$ is (the unique, even and verifying $\phi(0) = 1$ when $l = 0$) solution of the equation:

$$\begin{aligned}&\frac{d^2\phi}{dt^2} + \{(2\alpha+1)\coth t + (2\beta+1)\operatorname{th} t\}\frac{d\phi}{dt}\\ &\quad + \{-(2\alpha+l)l(\operatorname{sh} t)^{-2} + (2\beta+m)m(\operatorname{ch} t)^{-2} + \lambda^2 + (\alpha+\beta+1)^2\}\phi = 0\end{aligned} \tag{5.27}$$

The functions $\phi^{(\alpha,\beta)}_{\lambda,l,m}$ are called *modified (or associated) Jacobi functions.*

We shall express now the scalar components of $\Phi^p_q(\lambda,\cdot)$ in terms of Jacobi functions. Gaillard ([Gai86], Lemme fondamental) obtained equivalent expressions in terms of hypergeometric functions for $\Phi^p_p(\lambda,\cdot)$ by carrying out a calculation in polar coordinates. It had been also remarked by Camporesi and Higuchi ([CH94]) that some differential forms verifying particular differential equations (which are the analogue of the equations verified by our τ_p-spherical functions) are expressed by means of hypergeometric functions. In this article, the authors determine these eigenforms as the analytic continuation of forms verifying the same equations on the compact dual of $H^n(\mathbb{R})$ — that is, the sphere $\mathbb{S}^n$ —, and compute them by a calculation in polar coordinates.

Theorem 5.4. *Let p be generic. The scalar components $\varphi_{p-1}(\lambda,\cdot)$, $\varphi_p(\lambda,\cdot)$ of $\Phi = \Phi_p^p(\lambda,\cdot)$ can be expressed in terms of the Jacobi functions defined in (5.22) and (5.26):*

$$\varphi_{p-1}(\lambda,t) = \phi_\lambda^{(\frac{n}{2},-\frac{1}{2})}(t), \tag{5.28}$$

$$\varphi_p(\lambda,t) = \tfrac{n}{n-p}\phi_\lambda^{(\frac{n}{2}-1,-\frac{1}{2})}(t) - \tfrac{p}{n-p}\phi_{\lambda,0,1}^{(\frac{n}{2},-\frac{3}{2})}(t). \tag{5.29}$$

Similarly, the scalar components $\varphi_{p-1}(\lambda,\cdot)$, $\varphi_p(\lambda,\cdot)$ of $\Phi = \Phi_{p-1}^p(\lambda,\cdot)$ are exactly:

$$\varphi_{p-1}(\lambda,t) = \tfrac{n}{p}\phi_\lambda^{(\frac{n}{2}-1,-\frac{1}{2})}(t) - \tfrac{n-p}{p}\phi_{\lambda,0,1}^{(\frac{n}{2},-\frac{3}{2})}(t), \tag{5.30}$$

$$\varphi_p(\lambda,t) = \phi_\lambda^{(\frac{n}{2},-\frac{1}{2})}(t). \tag{5.31}$$

Proof : let us show first (5.31) and (5.30). The differential equation (5.16) can be written as

$$2p(\operatorname{sh} t)^{-1}(\coth t)\varphi_{p-1}(\lambda,t) = 2(\coth t)\frac{d}{dt}\varphi_p(\lambda,t) + 2p(1+(\operatorname{sh} t)^{-2})\varphi_p(\lambda,t).$$

Substituting this expression in (5.12) and replacing λ^2 by $\mu^2 + n - 2p$:

$$\frac{d^2}{dt^2}\varphi_p(\mu,t) + (n+1)(\coth t)\frac{d}{dt}\varphi_p(\mu,t) + \{\mu^2 + n + (\tfrac{n-1}{2})^2\}\varphi_p(\mu,t) = 0,$$

that is to say:

$$\frac{d^2}{dt^2}\varphi_p(\mu,t) + (n+1)(\coth t)\frac{d}{dt}\varphi_p(\mu,t) + \{\mu^2 + (\rho+1)^2\}\varphi_p(\mu,t) = 0,$$

which is exactly the equation (5.25) with $\alpha = \frac{n}{2}$, $\beta = -\frac{1}{2}$ and $\lambda = \mu$. Since $\varphi_p(\lambda,\cdot)$ is C^∞, even and verifies $\varphi_p(\lambda,0) = 1$, we have shown (5.31). To deduce the expression of φ_{p-1}, we need two 'technical' results.

Lemma 5.5. *In the particular case $\beta = -\frac{1}{2}$, one has the following formulæ:*

(a) $$\left(-\frac{1}{\operatorname{sh} t}\frac{d}{dt}\right)\phi_\lambda^{(\alpha,-\frac{1}{2})}(t) = c_\lambda^{(\alpha)}\phi_\lambda^{(\alpha+1,-\frac{1}{2})}(t), \qquad \text{where } c_\lambda^{(\alpha)} = \tfrac{(\alpha+\frac{1}{2})^2+\lambda^2}{2(\alpha+1)}.$$

(b) $$\frac{1}{2(\alpha+1)}(\operatorname{sh} t)\frac{d}{dt}\phi_\lambda^{(\alpha+1,-\frac{1}{2})}(t) = -(\operatorname{ch} t)\,\phi_\lambda^{(\alpha+1,-\frac{1}{2})}(t) + \phi_\lambda^{(\alpha,-\frac{1}{2})}(t).$$

Proof of (a)*:* consider the 'shifted' Jacobi operator (5.24)

$$\widetilde{L}_{\alpha,-\frac{1}{2}} = \frac{d^2}{dt^2} + (2\alpha+1)(\coth t)\frac{d}{dt} + \left(\alpha + \frac{1}{2}\right)^2.$$

$\phi_\lambda^{(\alpha,-\frac{1}{2})}$ is then the unique C^∞ solution on $\mathbb{R}$ of the equation

$$\widetilde{L}_{\alpha,-\frac{1}{2}}\,\phi_\lambda^{(\alpha,-\frac{1}{2})}(t) = -\lambda^2\phi_\lambda^{(\alpha,-\frac{1}{2})}(t),$$

with the additional conditions of being even and of verifying $\phi_\lambda^{(\alpha,-\frac{1}{2})}(0) = 1$. From the 'shift' relation

$$\left(-\tfrac{1}{\operatorname{sh} t}\tfrac{d}{dt}\right)\circ\widetilde{L}_{\alpha,-\frac{1}{2}} = \widetilde{L}_{\alpha+1,-\frac{1}{2}}\circ\left(-\tfrac{1}{\operatorname{sh} t}\tfrac{d}{dt}\right),$$

we get the identity:

$$\left(-\tfrac{1}{\operatorname{sh} t}\tfrac{d}{dt}\right)\phi_\lambda^{(\alpha,-\frac{1}{2})}(t) = c_\lambda^{(\alpha)}\phi_\lambda^{(\alpha+1,-\frac{1}{2})}(t),$$

where

$$\begin{aligned} c_\lambda^{(\alpha)} &= \left(-\tfrac{1}{\operatorname{sh} t}\tfrac{d}{dt}\right)\Big|_{t=0}\,\phi_\lambda^{(\alpha,-\frac{1}{2})}(t) \\ &= 2\operatorname{ch}(0)\tfrac{d}{dz}\Big|_{z=0}\,{}_2F_1\left(\tfrac{1}{2}\{\alpha+\tfrac{1}{2}+i\lambda\},\tfrac{1}{2}\{\alpha+\tfrac{1}{2}-i\lambda\};\alpha+1;z\right) \\ &= \tfrac{(\alpha+\frac{1}{2})^2+\lambda^2}{2(\alpha+1)}. \end{aligned}$$

The last equality follows from the elementary formula

$$\tfrac{d}{dz}\,{}_2F_1(a,b;c;z) = \tfrac{ab}{c}\,{}_2F_1(a+1,b+1;c+1;z).$$

Proof of (b): it suffices to differentiate both sides of (a) with respect to t. ✓

We come back to the proof of Theorem 5.4. Starting from (5.16) and, knowing that $\varphi_p(\lambda,\cdot) = \phi_\lambda^{(\frac{n}{2},-\frac{1}{2})}$, we use formula (b) in the previous lemma with $\alpha = \frac{n}{2}-1$. Then

$$\begin{aligned} \varphi_{p-1}(\lambda,t) &= (\operatorname{ch} t)\phi_\lambda^{(\frac{n}{2},-\frac{1}{2})}(t) + \tfrac{1}{p}(\operatorname{sh} t)\frac{d}{dt}\phi_\lambda^{(\frac{n}{2},-\frac{1}{2})}(t) \\ &= (\operatorname{ch} t)\phi_\lambda^{(\frac{n}{2},-\frac{1}{2})}(t) + \tfrac{1}{p}(\operatorname{sh} t)\Big\{-n(\coth t)\phi_\lambda^{(\frac{n}{2},-\frac{1}{2})}(t) + \frac{n}{\operatorname{sh} t}\phi_\lambda^{(\frac{n}{2}-1,-\frac{1}{2})}(t)\Big\} \\ &= \tfrac{n}{p}\phi_\lambda^{(\frac{n}{2}-1,-\frac{1}{2})}(t) - \tfrac{n-p}{p}(\operatorname{ch} t)\phi_\lambda^{(\frac{n}{2},-\frac{1}{2})}(t). \end{aligned}$$

This proves (5.30). One can use the same argument to prove (5.28) and (5.29) or, more simply, proceed by duality. ✓

Corollary 5.6. *Let p be generic. Put $\Delta_1 = d^*d$ and $\Delta_2 = dd^*$. Then:*

(i) *the following conditions are equivalent for a normalized τ_p-radial function:*
- $\Phi(x) = \Phi_p^p(\lambda, x)$, *resp.* $\Phi(x) = \Phi_{p-1}^p(\lambda, x)$ *for all $x \in G$ and for some $\lambda \in \mathbb{C}$;*
- $\Delta\Phi = \{\lambda^2 + (\rho - p)^2\}\Phi$ *and* $d^*\Phi = 0$, *resp.* $\Delta\Phi = \{\lambda^2 + [\rho - (p-1)]^2\}\Phi$ *and* $d\Phi = 0$*;*
- $\Delta_1\Phi = \{\lambda^2 + (\rho - p)^2\}\Phi$ *and* $\Delta_2\Phi = 0$, *resp.* $\Delta_2\Phi = \{\lambda^2 + [\rho - (p-1)]^2\}\Phi$ *and* $\Delta_1\Phi = 0$*;*

(ii) $\Sigma(G, K, \tau_p, \tau_p) = \{\Phi_q^p(\lambda, \cdot) : q = p-1, p \text{ and } \lambda \in \mathbb{C}/\pm 1\}$.

(iii) *there are no nonzero τ_p-radial functions which are harmonic and L^2 in the generic case.*

Proof of (i)*:* according to Theorem 5.4, the scalar components of Φ_q^p are uniquely determined as solutions of certain differential equations coming from the differential equations given in the second condition. The third condition is a rewriting with the differential operators preserving the degree p (see Corollary 4.7).

Proof of (ii)*:* it is the consequence of a general result (see Proposition B.15 and the following remark), but it has a simple proof in our setting, which is based on the following result.

Lemma 5.7. *Each element $D \in \mathbb{D}(G, K, \tau_p)$ can be written in a unique way*

$$D = a\,\mathrm{Id} + \sum_{j>0} b_j \Delta_1^j + \sum_{j>0} c_j \Delta_2^j.$$

Moreover, the characters of the algebra $\mathbb{D}(G, K, \tau_p)$ are of two types:

$$\mathcal{X}_1(D) = a + \sum_{j>0} b_j \nu_1^j, \qquad \mathcal{X}_2(D) = a + \sum_{j>0} c_j \nu_2^j,$$

where ν_1 and ν_2 are two arbitrary scalars.

Proof : we know (Corollary 2.2) that $\mathbb{D}(G, K, \tau_p)$ is generated by Δ_1 and Δ_2. Since

$$\Delta_1\Delta_2 = 0 = \Delta_2\Delta_1 \tag{5.32}$$

is the only relation between the generators, each element $D \in \mathbb{D}(G, K, \tau_p)$ is expressed as claimed, and each character $\mathcal{X}$ of $\mathbb{D}(G, K, \tau_p)$ is of the form

$$\mathcal{X}(D) = a + \sum_{j>0} b_j \nu_1^j + \sum_{j>0} c_j \nu_2^j, \tag{5.33}$$

where $\nu_1 = \mathcal{X}(\Delta_1)$ and $\nu_2 = \mathcal{X}(\Delta_2)$ are scalars subjected to the sole condition (5.32), i.e. $\nu_1\nu_2 = 0$. Thus, one (at least) of the complex numbers ν_1, ν_2 is always zero in (5.33), and $\mathcal{X}$ can only be of the type $\mathcal{X}_1$ or $\mathcal{X}_2$. ✓

Combining Lemma 5.7 and Lemma B.13, one obtains that there are exactly two τ_p-spherical functions, which are necessarily $\Phi^p_p(\lambda,\cdot)$ and $\Phi^p_{p-1}(\lambda,\cdot)$. (By the way, notice that the corresponding characters are given by $\mathcal{X}(\Delta_1) = \lambda^2 + (\rho - p)^2$, resp. $\mathcal{X}(\Delta_2) = \lambda^2 + [\rho - (p-1)]^2$.)

Proof of (iii)*:* according to (i) above, the harmonic τ_p-radial functions have to be found out among the functions $\Phi^p_q(\pm i(\rho - q),\cdot)$ and it is natural to wonder if they could be L^2.

The answer follows from the asymptotic behaviour of the Jacobi functions ([Koo84], relation (2.19)):

$$\phi_\lambda^{(\frac{n}{2},-\frac{1}{2})}(t) \underset{t\to+\infty}{\sim} c(\lambda)\, e^{(i\lambda-\frac{n+1}{2})t}, \quad \text{for } \operatorname{Im}(\lambda) < 0$$

(and the same behaviour with $-\lambda$ instead of λ when $\operatorname{Im}(\lambda) > 0$) [2]. As a consequence, the components φ_{p-1}, φ_p of $\Phi^p_p(\pm i(\rho - p),\cdot)$ behave at infinity as follows:

$$\varphi_{p-1}(t) \underset{t\to+\infty}{\sim} \text{cst } e^{-(p+1)t},$$
$$\varphi_p(t) \underset{t\to+\infty}{\sim} \text{cst } e^{-pt},$$

and we have similar estimates for the components of $\Phi^p_{p-1}(\pm i(\rho - [p-1]),\cdot)$. Hence

$$\Phi^p_q(\pm i(\rho - q), a_t) \underset{t\to+\infty}{\sim} \text{cst } e^{-qt}.$$

Using (5.4), one gets then

$$\begin{aligned}\|\Phi^p_q(\pm i(\rho - q),\cdot)\|^2_{L^2} &\sim \text{cst} \lim_{s\to+\infty} \int_0^s dt\, e^{(n-1)t} e^{-2qt}\\ &\sim \text{cst} \lim_{s\to+\infty} \int_0^s dt\, e^{2(\rho-q)t}.\end{aligned}$$

Therefore, $\|\Phi^p_q(\pm i(\rho - q),\cdot)\|^2_{L^2} < +\infty$ if and only if $q > \rho = \frac{n-1}{2}$ and — by duality — $q < \frac{n+1}{2}$, which excludes only the case n even and $q = n/2$, that we will deal with later.

This concludes the proof of Corollary 5.6. ✓

[2] The function $\lambda \mapsto c(\lambda)$ will be defined in (6.11).

Special case $p = \frac{n-1}{2}$

Now, fix $p = \frac{n-1}{2}$. Similarly to the generic case, and with the notations of Section 4, we define, for $\lambda \in \mathbb{C}$:

$$\Phi^p_{p-1}(\lambda, \cdot) : G \longrightarrow \mathrm{End}(\wedge^p \mathbb{C}^n),$$
$$\Phi^p_{p-1}(\lambda, x) = P^p_{p-1} \circ \pi_{\sigma_{p-1},\lambda}(x^{-1}) \circ J^p_{p-1}, \tag{5.34}$$
$$\text{and } \Phi^p_{p,\pm}(\lambda, \cdot) : G \longrightarrow \mathrm{End}(\wedge^p \mathbb{C}^n),$$
$$\Phi^p_{p,\pm}(\lambda, x) = P^p_{p,\pm} \circ \pi_{\sigma_p^\pm,\lambda}(x^{-1}) \circ J^p_{p,\pm}. \tag{5.35}$$

The functions $\Phi^p_{p-1}(\lambda, \cdot)$, $\Phi^p_{p,\pm}(\lambda, \cdot)$ are τ_p-spherical by Proposition B.9, and they verify — as Poisson transforms — some differential equations given in Corollary 4.7. Precisely:

$$\Phi = \Phi^p_{p-1}(\lambda, \cdot) \text{ verifies } \quad \begin{cases} \Delta\Phi = (\lambda^2 + 1)\Phi, & (5.36) \\ d\Phi = 0, & (5.37) \end{cases}$$

$$\text{and } \Phi = \Phi^p_{p,\pm}(\lambda, \cdot) \text{ verifies } \quad \begin{cases} d{*}\Phi = 0, & (5.38) \\ {*}d\Phi = \pm i^{p^2-1}\lambda\, \Phi, & (5.39) \\ \Delta\Phi = \lambda^2\, \Phi. & (5.40) \end{cases}$$

(Here, certain equations coming from Corollary 4.7 have been replaced by simpler and equivalent equations.) (5.40) is given for information only, since it is obviously implied by the previous two.

Remarks:

1. Proposition B.14 gives the following integral expressions:

$$\Phi^p_{p-1}(\lambda, x) = 2 \int_K dk\, e^{-(i\lambda+\rho)H(xk)} \tau_p(k) \circ P_{\sigma_{p-1}} \circ \tau_p(\underline{k}(xk))^{-1},$$
$$\Phi^p_{p,\pm}(\lambda, x) = \tfrac{4n}{n+1} \int_K dk\, e^{-(i\lambda+\rho)H(xk)} \tau_p(k) \circ P_{\sigma_p^\pm} \circ \tau_p(\underline{k}(xk))^{-1}.$$

2. The unitarity of the occuring principal series implies also the relation $\Phi^p_{q(,\pm)}(\lambda, x)^* = \Phi^p_{q(,\pm)}(\bar{\lambda}, x^{-1})$, but the action of the Weyl group gives now the relation $\Phi^p_{p,\pm}(\lambda, x) = \Phi^p_{p,\mp}(-\lambda, x)$ — while $\Phi^p_{p-1}(\cdot, x)$ is still an even function.

Our aim is again to make explicit the scalar components of the τ_p-spherical functions defined above. The difference with the generic case is that, *a priori*, the restrictions to A of the τ_p-spherical functions do not verify the condition $\Phi(a_{-t}) = \Phi(a_t)$, i.e. their scalar components are not necessarily all even (see Section 5.1). The following result shows that it is however the case for $\Phi^p_{p-1}(\lambda, \cdot)$.

Lemma 5.8. *Let $p = \frac{n-1}{2}$. The τ_p-spherical function $\Phi = \Phi^p_{p-1}(\lambda, \cdot)$ verifies*

$$\Phi(\lambda, a_{-t}) = \Phi(\lambda, a_t).$$

In particular, its three scalar components $\varphi_{p-1}(\lambda, \cdot)$, $\varphi^+_p(\lambda, \cdot)$, $\varphi^-_p(\lambda, \cdot)$ are even in the variable t and, therefore, $\varphi^+_p = \varphi^-_p := \varphi_p$. Thus, the determination of the scalar components of $\Phi^p_{p-1}(\lambda, \cdot)$ can be carried out as in the generic case.

Proof : for an arbitrary $\lambda \in \mathbb{C}$, define

$$\Phi(x) = P^{p-1}_{p-1} \circ \pi_{\sigma_{p-1},\lambda}(x)^{-1} \circ J^p_{p-1} \in \mathrm{Hom}(\wedge^p \mathbb{C}^n, \wedge^{p-1}\mathbb{C}^n).$$

Then $\Phi(a_t)$ is an element of $\mathrm{Hom}_M(\wedge^p \mathbb{C}^n, \wedge^{p-1}\mathbb{C}^n)$, which is one dimensional and whose generator is the homomorphism:

$$e_1 \wedge \wedge^{p-1}\mathbb{C}^{n-1} \oplus \wedge^p_+ \mathbb{C}^{n-1} \oplus \wedge^p_- \mathbb{C}^{n-1} \to e_1 \wedge \wedge^{p-2}\mathbb{C}^{n-1} \oplus \wedge^{p-1}\mathbb{C}^{n-1},$$
$$e_1 \wedge \eta + \xi^+ + \xi^- \mapsto 0 + \eta.$$

Thus, $\Phi(a_t)\xi \neq 0$ if and only if $\xi = e_1 \wedge \eta \in e_1 \wedge \wedge^{p-1}\mathbb{C}^{n-1}$. Moreover, for all η,

$$\begin{aligned}
\Phi(a_{-t})e_1 \wedge \eta &= \tau_{p-1}(m')^{-1}\Phi(a_t)\tau_p(m')e_1 \wedge \eta \\
&= -\tau_{p-1}(m')^{-1}\Phi(a_t)e_1 \wedge \tau_{p-1}(m')\eta \\
&= -\Phi(a_t)e_1 \wedge \eta.
\end{aligned}$$

(We used again the action of the nontrivial element of the Weyl group.) This shows that the function $t \mapsto \Phi(a_t)$ is odd. Furthermore,

$$\begin{aligned}
d\Phi(a_{-t})\eta &= d\Phi(\theta a_t)\eta \quad (\theta \text{ is the Cartan involution}) \\
&= \sum_{j=1}^n e_j \wedge \Phi(\theta a_t : e_j)\eta \quad ((e_j) \text{ is a basis of } \mathfrak{p}_\mathbb{C}) \\
&= -\sum_{j=1}^n e_j \wedge \Phi(\theta a_t : \theta e_j)\eta \\
&= \sum_{j=1}^n e_j \wedge \Phi(a_t : e_j)\eta \\
&= d\Phi(a_t)\eta.
\end{aligned}$$

But, according to (4.18),

$$d\Phi(x) = \sqrt{\tfrac{n-1}{n+3}}(1 - i\lambda)\Phi^p_{p-1}(\lambda, x),$$

which shows that $\Phi^p_{p-1}(\lambda, a_{-t}) = \Phi^p_{p-1}(\lambda, a_t)$ for $\lambda \neq -i$, and thus for all $\lambda \in \mathbb{C}$ by holomorphic continuation. (More generally, one can see that $\Phi^p_{p-1}(\lambda, \theta x) = \Phi^p_{p-1}(\lambda, x)$.)
This condition implies that all scalar components of $\Phi^p_{p-1}(\lambda,\cdot)$ are even (and not only $\varphi_{p-1}(\lambda,\cdot)$) and, since we know that $\varphi^+_p(\lambda,t) = \varphi^-_p(\lambda,-t)$, we are done. ✓

If the determination of the scalar components of $\Phi^p_{p-1}(\lambda,\cdot)$ is done as in the generic case, thanks to the identity $\varphi^+_p = \varphi^-_p$, it is not the case for the spherical function $\Phi^p_{p,\pm}(\lambda,\cdot)$, and the calculations are more delicate, as we shall see now.

Lemma 5.9. *Let $p = \frac{n-1}{2}$. Denote by φ_p the average of φ^+_p and φ^-_p (i.e. the even part of φ^+_p). Then:*

(i) *the equation (5.38) corresponds to the following equation for the scalar components $\varphi_{p-1}(\lambda,\cdot)$, $\varphi^+_p(\lambda,\cdot)$ and $\varphi^-_p(\lambda,\cdot)$ of $\Phi^p_{p,\pm}(\lambda,\cdot)$:*

$$\frac{d}{dt}\varphi_{p-1}(\lambda,t) + \tfrac{n+1}{2}(\coth t)\varphi_{p-1}(\lambda,t) = \tfrac{n+1}{2}(\operatorname{sh} t)^{-1}\varphi_p(\lambda,t); \tag{5.41}$$

(ii) *the equation (5.39) corresponds to the following equations for the scalar components $\varphi_{p-1}(\lambda,\cdot)$, $\varphi^+_p(\lambda,\cdot)$ and $\varphi^-_p(\lambda,\cdot)$ of $\Phi^p_{p,\pm}(\lambda,\cdot)$:*

$$\tfrac{1}{2}\{\varphi^+_p(\lambda,t) - \varphi^-_p(\lambda,t)\} = \mp i\,\tfrac{2\lambda}{n+1}(\operatorname{sh} t)\varphi_{p-1}(\lambda,t), \tag{5.42}$$

$$\frac{d}{dt}\varphi^+_p(\lambda,t) + \{\tfrac{n-1}{2}\coth t \pm i\lambda\}\varphi^+_p(\lambda,t) = \tfrac{n-1}{2}(\operatorname{sh} t)^{-1}\varphi_{p-1}(\lambda,t), \tag{5.43}$$

$$\frac{d}{dt}\varphi^-_p(\lambda,t) + \{\tfrac{n-1}{2}\coth t \mp i\lambda\}\varphi^-_p(\lambda,t) = \tfrac{n-1}{2}(\operatorname{sh} t)^{-1}\varphi_{p-1}(\lambda,t). \tag{5.44}$$

Proof of (i): set, for short, $\Phi(x) = \Phi^p_{p,\pm}(\lambda,x)$ and $\varphi(t) = \varphi(\lambda,t)$. By restricting to $x = a_t$,

$$\begin{aligned}(d{*}\Phi)(a_t)\xi = e_1 \wedge \frac{d}{dt}({*}\Phi)(a_t)\xi + (\operatorname{sh} t)^{-1}\sum_{j=2}^{n} e_j \wedge {*}\Phi(a_t)\tau_p(\widetilde{Z}_j)\xi \\ - (\coth t)\sum_{j=2}^{n} e_j \wedge \tau_{p+1}(\widetilde{Z}_j){*}\Phi(a_t)\xi.\end{aligned} \tag{5.45}$$

Since one has the expressions

$${*}\Phi(a_t)\xi = \begin{cases} \varphi_{p-1}(t){*}\eta & \text{if } \xi = e_1 \wedge \eta \in e_1 \wedge \wedge^{p-1}\mathbb{C}^{n-1}, \\ \pm i^{p(p+2)}\varphi^\pm_p(t) e_1 \wedge \xi & \text{if } \xi \in \wedge^p_\pm \mathbb{C}^{n-1}, \end{cases}$$

one finds that (5.45) gives '$0=0$' on $\xi \in \wedge^p_\pm \mathbb{C}^{n-1}$. On the contrary, if $\xi = e_1 \wedge \eta$,

$$(d{*}\Phi)(a_t)\xi = \frac{d}{dt}\varphi_{p-1}(t)e_1 \wedge {*}\eta - (\operatorname{sh} t)^{-1}\sum_{j=2}^{n} e_j \wedge {*}\Phi(a_t)e_j \wedge \eta \qquad (5.46)$$

$$- (\coth t)\sum_{j=2}^{n} \varphi_{p-1}(t)e_j \wedge \tau_{p+1}(\widetilde{Z}_j){*}\eta.$$

Simplify the second term in the right-hand side of (5.46):

$${*}\Phi(a_t)e_j \wedge \eta = i^{p(p+2)}\varphi_p^+(t)e_1 \wedge (e_j \wedge \eta)_+ - i^{p(p+2)}\varphi_p^-(t)e_1 \wedge (e_j \wedge \eta)_-,$$

where

$$(e_j \wedge \eta)_\pm = \tfrac12 e_j \wedge \eta \pm \tfrac12 i^{p(p+2)}{*}(e_j \wedge \eta)$$
$$= \tfrac12 e_j \wedge \eta \pm \tfrac12 i^{p^2+2} i_{e_j}({*}\eta).$$

Hence

$$e_j \wedge {*}\Phi(a_t)e_j \wedge \eta = \tfrac12 i^{p(p+2)}\{\varphi_p^+(t) - \varphi_p^-(t)\}e_j \wedge e_1 \wedge e_j \wedge \eta$$
$$- \tfrac12\{\varphi_p^+(t) + \varphi_p^-(t)\}e_j \wedge e_1 \wedge i_{e_j}({*}\eta),$$

so that

$$\sum_{j=2}^{n} e_j \wedge {*}\Phi(a_t)e_j \wedge \eta = \tfrac{n+1}{2}\varphi_p(t)e_1 \wedge {*}\eta.$$

Simplify the third term in the right-hand side of (5.46):

$$e_j \wedge \tau_{p+1}(\widetilde{Z}_j){*}\eta = -e_j \wedge e_j \wedge i_{e_1}({*}\eta) + e_j \wedge e_1 \wedge i_{e_j}({*}\eta),$$

which implies

$$\sum_{j=2}^{n} e_j \tau_{p+1}(\widetilde{Z}_j){*}\eta = -\tfrac{n+1}{2} e_1 \wedge {*}\eta.$$

To sum up, if $\xi = e_1 \wedge \eta$,

$$(d{*}\Phi)(a_t)\xi = \{\tfrac{d}{dt}\varphi_{p-1}(t) - \tfrac{n+1}{2}(\operatorname{sh} t)^{-1}\varphi_p(t) + \tfrac{n+1}{2}(\coth t)\varphi_{p-1}(t)\}e_1 \wedge {*}\eta, \quad (5.47)$$

and one gets (5.41).

Proof of (ii)*:* to make easier the calculations, consider the equation

$$d\Phi = \pm i^{p^2-1}\lambda\, {*}\Phi, \qquad (5.48)$$

which is equivalent to (5.39). As seen in the generic case,

$$d\Phi(a_t)\xi = e_1 \wedge \frac{d}{dt}\Phi(a_t)\xi + (\operatorname{sh} t)^{-1}\sum_{j=2}^{n} e_j \wedge \Phi(a_t)\tau_p(\widetilde{Z}_j)\xi \qquad (5.49)$$
$$- (\coth t)\sum_{j=2}^{n} e_j \wedge \tau_p(\widetilde{Z}_j)\Phi(a_t)\xi.$$

Consider first the case where $\xi = e_1 \wedge \eta \in e_1 \wedge \wedge^{p-1}\mathbb{C}^{n-1}$. It is easy to see that (5.49) gives:

$$d\Phi(a_t)\xi = -(\operatorname{sh} t)^{-1}\sum_{j=2}^{n} e_j \wedge \Phi(a_t)e_j \wedge \eta,$$

with

$$\Phi(a_t)e_j \wedge \eta = \varphi_p^+(t)(e_j \wedge \eta)_+ + \varphi_p^-(t)(e_j \wedge \eta)_-,$$

where, as in the proof of (i),

$$(e_j \wedge \eta)_\pm = \tfrac{1}{2}e_j \wedge \eta \pm \tfrac{1}{2}i^{p^2+2}i_{e_j}(*\eta),$$

so that

$$e_j \wedge \Phi(a_t)e_j \wedge \eta = i^{p^2+2}\,\tfrac{1}{2}\{\varphi_p^+(t) - \varphi_p^-(t)\}e_j \wedge i_{e_j}(*\eta).$$

Thus, (5.49) becomes:

$$d\Phi(a_t)\xi = \tfrac{n+1}{2}i^{p^2}(\operatorname{sh} t)^{-1}\,\tfrac{1}{2}\{\varphi_p^+(t) - \varphi_p^-(t)\}*\eta. \qquad (5.50)$$

And, since in the right hand-side of (5.48) one has

$$*\Phi(a_t)\xi = \varphi_{p-1}(t)*(e_1 \wedge \eta) = \varphi_{p-1}(t)*\eta,$$

one deduces (5.42).
Consider now the case where $\xi \in \wedge_\pm^p\mathbb{C}^{n-1}$. (5.49) becomes then:

$$d\Phi(a_t)\xi = \{\tfrac{d}{dt}\varphi_p^\pm(t) - \tfrac{n-1}{2}(\operatorname{sh} t)^{-1}\varphi_{p-1}(t) + \tfrac{n-1}{2}(\coth t)\varphi_p^\pm(t)\}e_1 \wedge \xi. \qquad (5.51)$$

Finally, as

$$*\Phi(a_t)\xi = \varphi_p^\pm(t)*\xi = \pm i^{p(p+2)}\varphi_p^\pm(t)e_1 \wedge \xi,$$

one obtains equations (5.43) and (5.44), concluding the proof of the lemma. ✓

Remarks:

1. Summing equations (5.43) and (5.44), and combining the result with (5.42), one obtains a differential equation relating φ_{p-1} and φ_p (as in the generic case):

$$\frac{d}{dt}\varphi_p(\lambda,t) + \tfrac{n-1}{2}(\coth t)\varphi_p(\lambda,t) = \{\tfrac{n-1}{2}(\operatorname{sh} t)^{-1} - \tfrac{2\lambda^2}{n+1}\operatorname{sh} t\}\varphi_{p-1}(\lambda,t). \qquad (5.52)$$

2. Substracting (5.44) to (5.43) and combining the result with (5.42), one finds again (5.41).

Using the two previous lemmas, we can give the expression of the scalar components of $\Phi^p_{p-1}(\lambda,\cdot)$ and $\Phi^p_{p,\pm}(\lambda,\cdot)$ in terms of (modified) Jacobi functions.

Theorem 5.10. *Let $p = \frac{n-1}{2}$. The scalar components $\varphi_{p-1}(\lambda,\cdot)$, $\varphi_p^+(\lambda,\cdot)$, $\varphi_p^-(\lambda,\cdot)$ of $\Phi = \Phi^p_{p-1}(\lambda,\cdot)$ can be expressed in terms of the Jacobi functions defined by (5.22) and (5.26):*

$$\varphi_{p-1}(\lambda,t) = \tfrac{2n}{n-1}\phi_\lambda^{(\frac{n}{2}-1,-\frac12)}(t) - \tfrac{n+1}{n-1}\phi_{\lambda,0,1}^{(\frac{n}{2},-\frac32)}(t), \qquad (5.53)$$

$$\varphi_p^\pm(\lambda,t) = \phi_\lambda^{(\frac{n}{2},-\frac12)}(t). \qquad (5.54)$$

The scalar components $\varphi_{p-1}(\lambda,\cdot)$, $\varphi_p^+(\lambda,\cdot)$, $\varphi_p^-(\lambda,\cdot)$ of $\Phi = \Phi^p_{p,\pm}(\lambda,\cdot)$ can be expressed in a similar way:

$$\varphi_{p-1}(\lambda,t) = \phi_\lambda^{(\frac{n}{2},-\frac12)}(t), \qquad (5.55)$$

$$\varphi_p^+(\lambda,t) = \tfrac{2n}{n+1}\phi_\lambda^{(\frac{n}{2}-1,-\frac12)}(t) - \tfrac{n-1}{n+1}\phi_{\lambda,0,1}^{(\frac{n}{2},-\frac32)}(t) \mp i\tfrac{2\lambda}{n+1}\phi_{\lambda,1,0}^{(\frac{n}{2}-1,-\frac12)}(t), \qquad (5.56)$$

$$\varphi_p^-(\lambda,t) = \tfrac{2n}{n+1}\phi_\lambda^{(\frac{n}{2}-1,-\frac12)}(t) - \tfrac{n-1}{n+1}\phi_{\lambda,0,1}^{(\frac{n}{2},-\frac32)}(t) \pm i\tfrac{2\lambda}{n+1}\phi_{\lambda,1,0}^{(\frac{n}{2}-1,-\frac12)}(t). \qquad (5.57)$$

Proof : the components of $\Phi^p_{p-1}(\lambda,\cdot)$ are determined exactly as in the generic case (Theorem 5.5).

For $\Phi^p_{p,\pm}(\lambda,\cdot)$, one replaces φ_p in (5.52) by its expression given in (5.41). One then obtains immediatly (5.55), and can deduce (as in the generic case) the expression of φ_p:

$$\varphi_p(\lambda,t) = \tfrac{2n}{n+1}\phi_\lambda^{(\frac{n}{2}-1,-\frac12)}(t) - \tfrac{n-1}{n+1}\phi_{\lambda,0,1}^{(\frac{n}{2},-\frac32)}(t). \qquad (5.58)$$

As φ_p is the even part of φ_p^+, we are just left with the determination of the odd part, which we get from (5.42), since φ_{p-1} is known. This argument gives (5.56), which is equivalent to (5.57) by using the identity $\varphi_p^-(\lambda,t) = \varphi_p^+(\lambda,-t)$. ✓

The following result is obtained as in the generic case.

Corollary 5.11. *Let* $p = \frac{n-1}{2}$. *Then:*

(i) *the following conditions are equivalent for a normalized* τ_p*-radial function:*
- $\Phi(x) = \Phi^p_{p,\pm}(\lambda, x)$, *resp.* $\Phi(x) = \Phi^p_{p-1}(\lambda, x)$ *for all* $x \in G$ *and for some* $\lambda \in \mathbb{C}$;
- $d\Phi = \pm i^{p^2-1}\lambda * \Phi$ *and* $d^*\Phi = 0$, *resp.* $\Delta\Phi = (\lambda^2 + 1)\Phi$ *and* $d\Phi = 0$;
- $*d\Phi = \pm i^{p^2-1}\lambda\,\Phi$ *and* $dd^*\Phi = 0$, *resp.* $dd^*\Phi = (\lambda^2 + 1)\Phi$ *and* $*d\Phi = 0$;

(ii) $\Sigma(G, K, \tau_p, \tau_p) = \{\Phi^p_{p-1}(\lambda, \cdot), \Phi^p_{p,\pm}(\lambda, \cdot) : \lambda \in \mathbb{C}/\pm 1\}$.

(iii) *there are no nonzero* τ_p*-radial functions which are harmonic and* L^2 *in the special case* $p = \frac{n-1}{2}$.

Special case $p = \frac{n}{2}$

In the sequel, σ will denote either σ_{p-1} or σ_p, since these representations are equivalent when $p = n/2$. Imitating again (5.6), we put, for $\lambda \in \mathbb{C}$:

$$\begin{aligned} &\Phi^\pm(\lambda, \cdot) : G \longrightarrow \mathrm{End}(\wedge^{\frac{n}{2}}_{\pm}\mathbb{C}^n), \\ &\Phi^\pm(x) = P^\pm \circ \pi_{\sigma,\lambda}(x^{-1}) \circ J^\pm, \end{aligned} \tag{5.59}$$

where $J^\pm = J^{\frac{n}{2},\pm}_{\frac{n}{2}(-1)}$ is the isometric embedding from $\wedge^{\frac{n}{2}}_{\pm}\mathbb{C}^n$ into $L^2(K, M, \sigma)$ defined by (4.6) and $P^\pm = (J^\pm)^*$. The function $\Phi^\pm(\lambda, \cdot)$ is then $\tau^\pm_{\frac{n}{2}}$-spherical and, since $\Phi^\pm(\lambda, e) = \mathrm{Id}$, it is the unique solution of the differential equation

$$\Delta\Phi = (\lambda^2 + \tfrac{1}{4})\Phi. \tag{5.60}$$

(See Corollary 4.7, equation (4.42).)

REMARKS:

1. As in the other cases, one has the integral formula :

$$\Phi^\pm(\lambda, x) = 2\int_K dk\, e^{-(i\lambda+\rho)H(xk)}\tau^\pm(k) \circ P_\sigma \circ \tau^\pm(\underline{k}(xk)^{-1}).$$

2. Again, one has the identities $\Phi^\pm(\lambda, x)^* = \Phi^\pm(\bar{\lambda}, x^{-1})$ and $\Phi^\pm(\lambda, x) = \Phi^\pm(-\lambda, x)$.

Denote by $\varphi^\pm(\lambda, \cdot)$ the unique scalar component of $\Phi^\pm(\lambda, \cdot)$. It is a normalized $(\varphi^\pm(\lambda, 0) = 1)$ even solution of the differential equation:

$$\frac{d^2}{dt^2}\phi(t) + (n-1)(\coth t)\frac{d}{dt}\phi(t) + \frac{n}{\mathrm{sh}^2\, t}(\mathrm{ch}\, t - 1)\phi(t) + (\lambda^2 + \rho^2)\phi(t) = 0, \tag{5.61}$$

which comes from (5.60) (the proof is similar to the one in Lemma 5.2). This is why we easily obtain the following statement.

Theorem 5.12. *Let $p = \frac{n}{2}$. The scalar component $\varphi^{\pm}(\lambda,\cdot)$ of $\Phi^{\pm}(\lambda,\cdot)$ is a modified Jacobi function (5.26):*

$$\varphi^{+}(\lambda,t) = \varphi^{-}(\lambda,t) = \phi_{2\lambda,0,2}^{(\frac{n}{2}-1,\frac{n}{2}-1)}(\tfrac{t}{2}) = (\operatorname{ch}\tfrac{t}{2})^2 \phi_{2\lambda}^{(\frac{n}{2}-1,\frac{n}{2}+1)}(\tfrac{t}{2}).$$

Proof : it suffices to put $u = t/2$ in (5.61); after simplification, one obtains:

$$\left\{\frac{d^2}{du^2} + (n-1)(\coth u + \operatorname{th} u)\frac{d}{du} + \frac{2n}{\operatorname{ch}^2 u} + (2\lambda)^2 + (n-1)^2\right\}\phi(2u) = 0,$$

which is precisely the equation we are looking for (see (5.27)). ✓

Corollary 5.13. *Let $p = \frac{n}{2}$. Then:*

(i) *the following conditions are equivalent for a normalized $\tau_p^{\pm}$-radial function:*
- $\Phi(x) = \Phi^{\pm}(\lambda,x)$ *for all $x \in G$ and for some $\lambda \in \mathbb{C}$;*
- $\Delta\Phi = (\lambda^2 + \frac{1}{4})\Phi$*;*
- $\Phi(a_t) = \phi_{2\lambda,0,2}^{(\frac{n}{2}-1,\frac{n}{2}-1)}(\frac{t}{2})\,\mathrm{Id}$ *for all $t \in \mathbb{R}$;*

(ii) $\Sigma(G,K,\tau_p^{\pm},\tau_p^{\pm}) = \{\Phi^{\pm}(\lambda,\cdot) : \lambda \in \mathbb{C}/\pm 1\}$.

(iii) *the function $\Phi^{\pm}(\frac{i}{2},\cdot) = \Phi^{\pm}(-\frac{i}{2},\cdot) \in \Sigma(G,K,\tau_p^{\pm},\tau_p^{\pm})$ is harmonic and L^2 (and even of Schwartz type [3]), and it is the only one.*

Proof of (iii)*:* according to (5.60), it is clear that $\Phi^{\pm}(\lambda,\cdot)$ is harmonic if and only if $\lambda = \pm\frac{i}{2}$. In this case, let us investigate the square-integrability.
By parity, we may suppose $\lambda = \frac{i}{2}$. Remind that

$$\|\Phi^{\pm}(\tfrac{i}{2},\cdot)\|_{L^2}^2 = \tfrac{1}{2}\, C_n^{n/2} \int_0^{\infty} dt\,(2\operatorname{sh} t)^{n-1} |\varphi^{\pm}(t)|^2,$$

with $\varphi^{\pm}(t) = (\operatorname{ch}\frac{t}{2})^2 \phi_i^{(\frac{n}{2}-1,\frac{n}{2}+1)}(\frac{t}{2})$. A straightforward calculation shows that

$$\phi_i^{(\frac{n}{2}-1,\frac{n}{2}+1)}(\tfrac{t}{2}) = (\operatorname{ch}\tfrac{t}{2})^{-n-2},$$

hence $\varphi^{\pm}(t) \underset{t\to\infty}{\sim} \mathrm{cst}\; e^{-\frac{n}{2}t}$, which implies that $\|\Phi^{\pm}(\frac{i}{2},\cdot)\|_{L^2} < +\infty$. Besides, this estimate shows also that, for all $t \in \mathbb{R}$, $\phi^{\pm}(t)$ and all its derivatives are $O(e^{-(\rho+\frac{1}{2})t})$. The uniqueness assertion follows trivially from (i). ✓

[3] The definition of the Schwartz space for τ-radial functions is given in §6.1.

REMARKS:

1. Contrary to the generic case (Corollary 5.6), we have not used the behaviour for $\phi_i^{(\frac{n}{2}-1,\frac{n}{2}+1)}$ given by (2.19) of [Koo84], since it is erroneous for the values (α, β) we consider. In fact, when $\lambda \to i$ in the equality

$$\phi_\lambda(t) = c(\lambda)\Phi_\lambda(t) + c(-\lambda)\Phi_{-\lambda}(t),$$

both terms of the right-hand side contribute as $\frac{1}{2}(\operatorname{ch} t)^{-n-2}$.

2. $\|\Phi^\pm(\frac{i}{2},\cdot)\|_{L^2}$ can be easily evaluated:

$$\begin{aligned} \|\Phi^\pm(\tfrac{i}{2},\cdot)\|^2_{L^2} &= \tfrac{1}{2}\, C_n^{n/2} \int_0^\infty dt\,(2\operatorname{sh} t)^{n-1}(\operatorname{ch}\tfrac{t}{2})^{-2n} \\ &= 2^{2n-3}\, C_n^{n/2} \int_0^\infty dt\,(\operatorname{th}\tfrac{t}{2})^{n-1}(\operatorname{ch}\tfrac{t}{2})^{-2} \\ &= \frac{2^{2n-2}\, C_n^{n/2}}{n}. \end{aligned} \tag{5.62}$$

6 Fourier analysis on $\Gamma \wedge^p H^n(\mathbb{R})$

6.1 (Spherical) Fourier transform of radial functions

Generic case

Let $F \in C_c^\infty(G,K,\tau_p,\tau_p)$. The *($\tau_p$-spherical) Fourier transform* $\mathcal{H}(F)$ of F is the couple $(\mathcal{H}_{p-1}^p(F),\mathcal{H}_p^p(F))$ of functions defined on $\mathbb{C}$ by :

$$\begin{aligned}\mathcal{H}_q^p(F)(\lambda) &= \frac{1}{C_n^p}\int_G dx\ \mathrm{tr}\{F(x)\Phi_q^p(\lambda,x^{-1})\} \qquad (q=p,p-1)\\ &= \frac{1}{C_n^p}\left(F,\Phi_q^p(\lambda,\cdot^{-1})^*\right) \qquad (q=p,p-1). \end{aligned} \tag{6.1}$$

Recall that $\mathcal{H}$ is nothing else but the Gelfand transform of the commutative algebra $C_c^\infty(G,K,\tau_p,\tau_p)$ (see Appendix B).
The aim of this section is to show that, as in the scalar case ($p=0$), the spherical Fourier transform reduces essentially to a Jacobi transform. This will allow us to use the powerful Jacobi analysis developed by Flensted-Jensen and Koornwinder in the 1970's (we shall often refer to the survey [Koo84]), and notably the Paley-Wiener and Plancherel theorems.

Before stating next result, we recall that the *Jacobi transform* $\mathcal{J}^{(\alpha,\beta)}(f)$ of a complex-valued function f on $\mathbb{R}$ is defined by:

$$\mathcal{J}^{(\alpha,\beta)}(f)(\lambda) = \int_0^\infty dt\,\zeta_{\alpha,\beta}(t)\phi_\lambda^{(\alpha,\beta)}(t)f(t), \tag{6.2}$$

where the weight $\zeta_{\alpha,\beta}$ was defined in (5.23).

Proposition 6.1. *Let p be generic and let $F \in C_c^\infty(G,K,\tau_p,\tau_p)$ with scalar components f_{p-1}, f_p. Then:*

(i) *the spherical Fourier transforms can be expressed in terms of Jacobi transforms:*

$$\mathcal{H}_p^p(F)(\lambda) = -\frac{1}{4n}\mathcal{J}^{(\frac{n}{2},-\frac{1}{2})}\Big(\frac{1}{\operatorname{sh}t}\{f_p'(t)+p(\coth t)f_p(t)-\frac{p}{\operatorname{sh}t}f_{p-1}(t)\}\Big)(\lambda), \tag{6.3}$$

$$\begin{aligned}\mathcal{H}_{p-1}^p(F)(\lambda) = -\frac{1}{4n}\mathcal{J}^{(\frac{n}{2},-\frac{1}{2})}\Big(\frac{1}{\operatorname{sh}t}\{f_{p-1}'(t)+(n-p)(\coth t)f_{p-1}(t)\\ -\frac{n-p}{\operatorname{sh}t}f_p(t)\}\Big)(\lambda);\end{aligned} \tag{6.4}$$

(ii) $\mathcal{H}^p_p(F) \equiv 0$ *if and only if* $dF = 0$, *and in that case*

$$\mathcal{H}^p_{p-1}(F)(\lambda) = \frac{\lambda^2 + [\rho - (p-1)]^2}{4np} \mathcal{J}^{(\frac{n}{2}, -\frac{1}{2})}(f_p)(\lambda); \tag{6.5}$$

$\mathcal{H}^p_{p-1}(F) \equiv 0$ *if and only if* $d^*F = 0$, *and in that case*

$$\mathcal{H}^p_p(F)(\lambda) = \frac{\lambda^2 + (\rho - p)^2}{4n(n-p)} \mathcal{J}^{(\frac{n}{2}, -\frac{1}{2})}(f_{p-1})(\lambda). \tag{6.6}$$

Proof of (i)*:* let us prove for instance (6.4). According to (6.1), $\mathcal{H}^p_{p-1}(F)(\lambda) = \frac{1}{C^p_n} I(\lambda)$, where

$$I(\lambda) = \big(F, \Phi^p_{p-1}(\lambda, \cdot^{-1})^*\big) .$$

As in Section 5, let us denote by $\varphi_q(\lambda, \cdot)$ the scalar components of $\Phi^p_{p-1}(\lambda, \cdot)$. Remember that

$$\begin{aligned} \varphi_p(\lambda, t) &= \phi_\lambda^{(\frac{n}{2}, -\frac{1}{2})}(t), \\ \varphi_{p-1}(\lambda, t) &= \frac{\operatorname{sh} t}{p} \varphi'_p(\lambda, t) + (\operatorname{ch} t)\varphi_p(\lambda, t). \end{aligned}$$

(Here, φ'_q means $\frac{d}{dt}\varphi_q$.) Since these functions are even, (5.4) gives

$$\begin{aligned} I(\lambda) &= \int_0^\infty dt\, (2\operatorname{sh} t)^{n-1} \{C^{p-1}_{n-1} f_{p-1}(t) \varphi_{p-1}(\lambda, t) + C^p_{n-1} f_p(t) \varphi_p(\lambda, t)\} \\ &= \int_0^\infty dt\, (2\operatorname{sh} t)^{n-1} \{C^{p-1}_{n-1} f_{p-1}(t) [\tfrac{\operatorname{sh} t}{p} \varphi'_p(\lambda, t) + (\operatorname{ch} t)\varphi_p(\lambda, t)] \\ &\qquad\qquad + C^p_{n-1} f_p(t) \varphi_p(\lambda, t)\}. \end{aligned}$$

Now, let us remark that

$$\begin{aligned} &(2\operatorname{sh} t)^{n-1} [\tfrac{\operatorname{sh} t}{p} \varphi'_p(\lambda, t) + (\operatorname{ch} t)\varphi_p(\lambda, t)] \\ &\qquad = \tfrac{1}{2p} [(2\operatorname{sh} t)^n \varphi'_p(\lambda, t) + 2p(\operatorname{ch} t)(2\operatorname{sh} t)^{n-1} \varphi_p(\lambda, t)] \\ &\qquad = \tfrac{1}{2p} [\tfrac{d}{dt}((2\operatorname{sh} t)^n \varphi_p(\lambda, t)) - 2(n-p)(\operatorname{ch} t)(2\operatorname{sh} t)^{n-1} \varphi_p(\lambda, t)]. \end{aligned}$$

Hence

$$\begin{aligned} I(\lambda) = \tfrac{C^{p-1}_{n-1}}{2p} &\int_0^\infty dt\, \tfrac{d}{dt} [(2\operatorname{sh} t)^n \varphi_p(\lambda, t)] f_{p-1}(t) \\ - \tfrac{C^{p-1}_{n-1}}{2p} &\int_0^\infty dt\, (2\operatorname{sh} t)^{n-1}\, 2(n-p)(\operatorname{ch} t) \varphi_p(\lambda, t) f_{p-1}(t) \\ + C^p_{n-1} &\int_0^\infty dt\, (2\operatorname{sh} t)^{n-1} \varphi_p(\lambda, t) f_p(t). \end{aligned}$$

But, since

$$\int_0^\infty dt\, \tfrac{d}{dt}[(2\operatorname{sh} t)^n \varphi_p(\lambda,t)] f_{p-1}(t) = -\int_0^\infty dt\,(2\operatorname{sh} t)^n \varphi_p(\lambda,t)] f'_{p-1}(t),$$

we get

$$\begin{aligned} I(\lambda) &= \int_0^\infty dt\,(2\operatorname{sh} t)^{n+1}\varphi_p(\lambda,t)\Big[-\tfrac{C^{p-1}_{n-1}}{4p\operatorname{sh} t} f'_{p-1}(t) - \tfrac{C^{p-1}_{n-1}(n-p)\operatorname{ch} t}{4p(\operatorname{sh} t)^2} f_{p-1}(t) + \tfrac{C^p_{n-1}}{4(\operatorname{sh} t)^2} f_p\Big] \\ &= -\tfrac{C^p_{n-1}}{4}\int_0^\infty dt\,(2\operatorname{sh} t)^{n+1}\tfrac{\varphi_p(\lambda,t)}{\operatorname{sh} t}\Big[\tfrac{f'_{p-1}(t)}{n-p} + (\coth t) f_{p-1}(t) - \tfrac{f_p(t)}{\operatorname{sh} t}\Big]. \end{aligned}$$

Now, since $\varphi_p(\lambda,t) = \phi_\lambda^{(\frac{n}{2},-\frac12)}(t)$, dividing $I(\lambda)$ by C^p_n gives exactly (6.4).

Proof of (ii)*:* suppose now $dF = 0$. Then Lemma 5.3 can be applied to F, which means

$$\begin{aligned} f_{p-1} &= \tfrac{\operatorname{sh} t}{p} f'_p + (\operatorname{ch} t) f_p, \\ f'_{p-1} &= \tfrac{\operatorname{sh} t}{p}[f''_p + (p+1)(\coth t) f'_p + p\, f_p]. \end{aligned} \tag{6.7}$$

Hence

$$\begin{aligned} f'_{p-1}(t) &+ (n-p)(\coth t) f_{p-1}(t) - \tfrac{n-p}{\operatorname{sh} t} f_p(t) \\ &= \tfrac{\operatorname{sh} t}{p}[L\, f_p(t) + p(n+1-p) f_p(t)]. \end{aligned}$$

($L = L_{\frac{n}{2},-\frac12}$ is the Jacobi Laplacian defined in (5.24).) As a consequence,

$$\mathcal{H}^p_{p-1}(F)(\lambda) = -\tfrac{1}{4np}\int_0^\infty dt\,(2\operatorname{sh} t)^{n+1}[L\, f_p(t) + p(n+1-p) f_p(t)]\phi_\lambda^{(\frac{n}{2},-\frac12)}(t).$$

But L is self-adjoint for the scalar product $(f,g) = \int_0^\infty dt\,(2\operatorname{sh} t)^{n+1} f(t) g(t)$ and, furthermore,

$$\begin{aligned} [L + p(n+1-p)]\phi_\lambda^{(\frac{n}{2},-\frac12)} &= -[\lambda^2 + (\rho+1)^2 + p(n+1-p)]\phi_\lambda^{(\frac{n}{2},-\frac12)} \\ &= -[\lambda^2 + [\rho-(p-1)]^2]\phi_\lambda^{(\frac{n}{2},-\frac12)}. \end{aligned}$$

Thus we obtain (6.5). On the other hand, (6.7) shows also that $\mathcal{H}^p_p(F) \equiv 0$ if $dF = 0$. Conversely, suppose $\mathcal{H}^p_p(F) \equiv 0$. Since $\mathcal{H}^p_p$ is a Jacobi transform, which is injective ([Koo84], Theorem 2.1), the expression between parantheses in (6.3) vanishes, and this means exactly that $dF = 0$.

The case $d^*F = 0$ can be worked out similarly. ✓

We now deal with the problem of inverting $\mathcal{H}$. The first step will be to give an inversion formula for τ_p-radial functions which belong to a particular subspace of $C_c^\infty(G, K, \tau_p, \tau_p)$, namely to

$$V_p = d(C_c^\infty(G, K, \tau_p, \tau_{p-1})) \oplus d^*(C_c^\infty(G, K, \tau_p, \tau_{p+1})). \tag{6.8}$$

Indeed, if $F \in V_p$, the spherical Fourier transforms of F are given by formulæ (6.5) and (6.6). Thus, given $\mathcal{H}_{p-1}^p(F)$ and $\mathcal{H}_p^p(F)$, one obtains the scalar components f_{p-1} and f_p of F by inverting $\mathcal{J}^{(\frac{n}{2},-\frac{1}{2})}$. Precisely, using [Koo84], Theorem 2.3,

$$f_{p-1}(t) = \frac{1}{2\pi}\int_0^\infty \frac{4n(n-p)}{\lambda^2+(\rho-p)^2}\,\frac{d\lambda}{|c(\lambda)|^2}\mathcal{H}_p^p(F)(\lambda)\phi_\lambda^{(\frac{n}{2},-\frac{1}{2})}(t), \tag{6.9}$$

$$f_p(t) = \frac{1}{2\pi}\int_0^\infty \frac{4np}{\lambda^2+[\rho-(p-1)]^2}\,\frac{d\lambda}{|c(\lambda)|^2}\mathcal{H}_{p-1}^p(F)(\lambda)\phi_\lambda^{(\frac{n}{2},-\frac{1}{2})}(t), \tag{6.10}$$

where c denotes *Harish-Chandra's function* defined by formula (2.18) of [Koo84]:

$$\begin{aligned} c(\lambda) = c^{(\frac{n}{2},-\frac{1}{2})}(\lambda) &= 2^{\frac{n+1}{2}-i\lambda}\frac{\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{1}{2}(i\lambda+\frac{n+1}{2}))}\,\frac{\Gamma(i\lambda)}{\Gamma(\frac{1}{2}(i\lambda+\frac{n+3}{2}))} \\ &= \frac{2^n\Gamma(\frac{n}{2}+1)}{\sqrt{\pi}}\,\frac{\Gamma(i\lambda)}{\Gamma(i\lambda+\frac{n+1}{2})}. \end{aligned} \tag{6.11}$$

Define now the *Plancherel measures* as follows:

- for the component $\mathcal{H}_{p-1}^p$, we put:

$$d\nu_{p-1}(\lambda) = \frac{2}{\pi}\,\frac{np}{\lambda^2+[\rho-(p-1)]^2}\,\frac{d\lambda}{|c(\lambda)|^2}; \tag{6.12}$$

- for the component $\mathcal{H}_p^p$, we put:

$$d\nu_p(\lambda) = \frac{2}{\pi}\,\frac{n(n-p)}{\lambda^2+(\rho-p)^2}\,\frac{d\lambda}{|c(\lambda)|^2}. \tag{6.13}$$

In order to describe the singularities of the Plancherel measures, we need the explicit expression of $|c(\lambda)|^{-2}$, which is equal to $[c(\lambda)c(-\lambda)]^{-1}$ when λ is real. By standard calculations relying on well-known properties of the Γ function (see e.g. the details in the special case $p = \frac{n}{2}$), we derive from (6.11) the following expressions:

- when n is odd,

$$[c(\lambda)c(-\lambda)]^{-1} = \Big[\Big(\frac{n+1}{2}\Big)\Big(\frac{n+1}{2}+1\Big)\dots(n-1)n\Big]^{-2}\lambda^2\prod_{k=1}^{\frac{n-1}{2}}(\lambda^2+k^2).$$

$[c(\lambda)c(-\lambda)]^{-1}$ is then polynomial, hence entire.

- when n is even,

$$[c(\lambda)c(-\lambda)]^{-1} = 2^{-2n}\,[(\tfrac{n}{2})!]^{-2}\,\pi^2\,\frac{\operatorname{th}\pi\lambda}{\pi\lambda}\lambda^2\prod_{k=1}^{\frac{n}{2}}\Big(\lambda^2+(k-\tfrac{1}{2})^2\Big).$$

  Here, $[c(\lambda)c(-\lambda)]^{-1}$ is meromorphic, and its (simple) poles belong to the set $\{\pm i(\rho+1+k), k\in\mathbb{N}\}$.

From the previous expressions, we deduce the list of (simple) poles of $d\nu_{p-1}$:

- for n odd: ρ is an integer, and $1 \leqslant \rho-(p-1) \leqslant \frac{n-1}{2}$ (for $1 \leqslant p \leqslant \frac{n-1}{2}$), then $d\nu_{p-1}$ is polynomial;
- for n even: ρ is a half-integer, and $\frac{1}{2} \leqslant \rho-(p-1) \leqslant \frac{n-1}{2}$ (for $1 \leqslant p \leqslant \frac{n}{2}$), then the poles of $d\nu_{p-1}$ range through the set $\{\pm i(\rho+1+k), k\in\mathbb{N}\}\cup\{\pm i[\rho-(p-1)]\}$.

In the same way, we describe the singularities of $d\nu_p$:

- for n odd: ρ is an integer, and $0 \leqslant \rho-p \leqslant \frac{n-3}{2}$ (for $1 \leqslant p \leqslant \frac{n-1}{2}$), then $d\nu_p$ is polynomial.
- for n even: ρ is a half-integer, and $-\frac{1}{2} \leqslant \rho-p \leqslant \frac{n-3}{2}$ (for $1 \leqslant p \leqslant \frac{n}{2}$). The division by the quantity $\lambda^2+(\rho-p)^2$ cancels one of the factors in the product (when $k-\frac{1}{2}=\rho-p$) that does not compensate any more the singularities of

$$\tfrac{\operatorname{th}\pi\lambda}{\pi\lambda} = \tfrac{2}{\pi}\textstyle\sum_{k=1}^{\infty}\frac{1}{\lambda^2+(k-\frac{1}{2})^2}$$

  at $\lambda = \pm i(\rho-p)$. The (simple) poles of $d\nu_p$ then range through the set $\{\pm i(\rho+1+k), k\in\mathbb{N}\}\cup\{\pm i(\rho-p)\}$.

Remark: we recover (up to normalizing constants) the expressions of the Plancherel measures established in [CH94], as well as the ones we can deduce from [Ven93], Example 3.4.

Proposition 6.2 (Intermediate inversion formula). *Let p be generic. The spherical Fourier transform $\mathcal{H}=(\mathcal{H}^p_{p-1},\mathcal{H}^p_p)$ of τ_p-radial functions $F\in V_p$ is inverted by the following formula:*

$$F(x) = \int_0^\infty \big\{d\nu_{p-1}(\lambda)\,\mathcal{H}^p_{p-1}(F)(\lambda)\,\Phi^p_{p-1}(\lambda,x) + d\nu_p(\lambda)\,\mathcal{H}^p_p(F)(\lambda)\,\Phi^p_p(\lambda,x)\big\}. \quad (6.14)$$

Proof : it suffices to prove (6.14) for $F_{|A}$, or, equivalently, (6.9) and (6.10) for its scalar components f_{p-1} and f_p. Moreover, it suffices also to consider the following two cases.

- *First case:* $F = dH$

Then $\mathcal{H}_p^p(F) \equiv 0$. Since the component $\varphi_p(\lambda, \cdot)$ of $\Phi_{p-1}^p(\lambda, \cdot)$ is $\phi_\lambda^{(\frac{n}{2}, -\frac{1}{2})}$, we have formula (6.10). On the other hand, differentiating both sides of (6.10) with respect to t, and using the identities $dF = 0 = d\Phi_{p-1}^p$, we get:

$$f_p' = \tfrac{p}{\operatorname{sh} t} f_{p-1} - p(\coth t) f_p,$$
$$\varphi_p' = \tfrac{p}{\operatorname{sh} t} \varphi_{p-1} - p(\coth t) \varphi_p.$$

Now, replacing these expressions in the differentiated version of (6.10), and using again (6.10), we obtain precisely (6.9).

- *Second case:* $F = d^* H$

Then $\mathcal{H}_{p-1}^p(F) \equiv 0$. The argument is similar to the one used in the first case and leads to (6.9) and (6.10). ✓

Remark: our motivation to introduce the subspace V_p is that the inversion formula (6.14) can't be directly stated for $F \in C_c^\infty(G, K, \tau_p, \tau_p)$, for V_p is dense in $L^2(G, K, \tau_p, \tau_p)$ (for $p \neq \frac{n}{2}$, by a variant of Theorem 2.3), but not in $C_c^\infty(G, K, \tau_p, \tau_p)$. Actually, let us mention that the proof of the inversion formula for $p = 1$ in [BR89] (Theorem 7.1) is incomplete, precisely because the authors seem to have misapplied the Hodge-Kodaira Theorem to V_1.

Theorem 6.3 (Plancherel). *Let p be generic. Define the space*

$$L^2(\mathbb{R}; \mu_1, \mu_2)_{even} = L^2(\mathbb{R}, \mu_1)_{even} \oplus L^2(\mathbb{R}, \mu_2)_{even},$$

where

$$L^2(\mathbb{R}, \mu)_{even} = \{h : \mathbb{R} \to \mathbb{R} \text{ even and such that } \textstyle\int_0^\infty \mu(\lambda)\, |h(\lambda)|^2 < +\infty\}.$$

Then:

(i) *Plancherel formula: for* $F \in V_p$,

$$\begin{aligned}\|F\|^2_{L^2} &= \int_G dx\, \|F(x)\|^2_{HS} \\ &= \int_0^\infty dt\, (2\operatorname{sh} t)^{n-1} \big\{ C^{p-1}_{n-1} |f_{p-1}(t)|^2 + C^p_{n-1} |f_p(t)|^2 \big\} \\ &= C^p_n \int_0^\infty \big\{ d\nu_{p-1}(\lambda) |\mathcal{H}^p_{p-1}(F)(\lambda)|^2 + d\nu_p(\lambda) |\mathcal{H}^p_p(F)(\lambda)|^2 \big\}. \quad (6.15)\end{aligned}$$

(ii) *The spherical Fourier transform* $\mathcal{H}$ *of* τ_p*-radial functions extends to a bijective isometry from*

$$L^2(G, K, \tau_p, \tau_p) \simeq L^2(\mathbb{R}; C^{p-1}_{n-1}(2\operatorname{sh} t)^{n-1} dt, C^p_{n-1}(2\operatorname{sh} t)^{n-1} dt)_{even}$$

onto $L^2(\mathbb{R}; C^p_n\, d\nu_{p-1}, C^p_n\, d\nu_p)_{even}$.

Proof of (i): we use classical arguments. Put $F^*(x) = F(x^{-1})^*$; apply the inversion formula (6.14) to the function $F * F^*$ at $x = e$, and take traces.
On the one hand,

$$\operatorname{tr}(F * F^*)(e) = \int_G dx\ \operatorname{tr}\{F(x)F(x)^*\} = \|F\|^2_{L^2}.$$

On the other hand, since $\mathcal{H}^p_q(F_1 * F_2) = \mathcal{H}^p_q(F_1) \cdot \mathcal{H}^p_q(F_2)$ by definition,

$$\begin{aligned}(F * F^*)(e) = \int_0^\infty \big\{ & d\nu_{p-1}(\lambda) \mathcal{H}^p_{p-1}(F)(\lambda) \mathcal{H}^p_{p-1}(F^*)(\lambda)\, \mathrm{Id}_{\wedge^p \mathbb{C}^n} \\ & + d\nu_p(\lambda) \mathcal{H}^p_p(F)(\lambda) \mathcal{H}^p_p(F^*)(\lambda)\, \mathrm{Id}_{\wedge^p \mathbb{C}^n} \big\}.\end{aligned}$$

Moreover, as $\operatorname{tr}(A^*B^*) = \overline{\operatorname{tr}(AB)}$ and $\Phi^p_q(\lambda, x^{-1}) = \Phi^p_q(\bar\lambda, x)^*$,

$$\begin{aligned}C^p_n\, \mathcal{H}^p_q(F^*)(\lambda) &= \int_G dx\ \operatorname{tr}(F(x^{-1})^* \Phi^p_q(\lambda, x^{-1})) \\ &= \int_G dx\, \overline{\operatorname{tr}(F(x^{-1}) \Phi^p_q(\bar\lambda, x))} \\ &= \int_G dx\, \overline{\operatorname{tr}(F(x) \Phi^p_q(\bar\lambda, x^{-1}))} \\ &= C^p_n\, \overline{\mathcal{H}^p_q(F)(\bar\lambda)}.\end{aligned}$$

Hence, since $\lambda \in \mathbb{R}$,

$$\operatorname{tr}\{F * F^*(e)\} = C^p_n \int_0^\infty \big\{ d\nu_{p-1}(\lambda) |\mathcal{H}^p_{p-1}(F)(\lambda)|^2 + d\nu_p(\lambda) |\mathcal{H}^p_p(F)(\lambda)|^2 \big\}.$$

The intermediate formula is trivially obtained by using (5.4).

Proof of (ii): (i) implies that $\mathcal{H}$ is an isometry; the injectivity of $\mathcal{H}$ is obvious by Proposition 6.1, and to establish the surjectivity, one uses Proposition 6.1, the Paley-Wiener theorem for the Jacobi transform (see [Koo84], Theorem 2.1, recalled below), the density of $PW(\mathbb{C})$ into $L^2(\mathbb{R}, d\nu_q)$ and the density of V_p into $L^2(G, K, \tau_p, \tau_p)$ (Hodge-Kodaira Theorem, see Theorem 2.3). ✓

The last step consists in stating a Paley-Wiener theorem and an inversion formula for general functions $F \in C_c^\infty(G, K, \tau_p, \tau_p)$. These results will be proved simultaneously.

Theorem 6.4 (Inversion formula). *Let p be generic. The inversion formula (6.14) still holds for functions $F \in C_c^\infty(G, K, \tau_p, \tau_p)$.*

For $R > 0$, define the set $PW_R(\mathbb{C})_{even}$ of entire functions h on $\mathbb{C}$ which are even and verify the condition:

$$\forall N \in \mathbb{N},\ \exists C_N > 0 : \forall \lambda \in \mathbb{C},\ |h(\lambda)| \leqslant C_N (1 + |\lambda|)^{-N} e^{R|\operatorname{Im} \lambda|}.$$

Then the Paley-Wiener theorem for $\mathcal{J}$ asserts that $\mathcal{J}$ is a topological linear isomorphism [1] between $PW_R(\mathbb{C})_{even}$ and the space $C^\infty([-R, +R])_{even}$ of C^∞ even functions on $\mathbb{R}$ with support in $[-R, +R]$.

Now, for $R > 0$, we denote by $C_R^\infty(G, K, \tau_p, \tau_p)$ the subspace of functions in $C_c^\infty(G, K, \tau_p, \tau_p)$ whose support is included in the closed ball $\overline{B}(o, R)$ of the hyperbolic space $H^n(\mathbb{R})$ (recall that $d(k_1 a_t k_2, o) = t \geqslant 0$ in the Cartan decomposition of G, and $\operatorname{Supp}(F) \subset \overline{B}(o, R) \Leftrightarrow \operatorname{Supp}(F_{|A}) \subset [-R, +R]$).

Theorem 6.5 (Paley-Wiener). *Let p be generic. For $q = p - 1$ or p, put $\lambda_q := i(\rho - q)$ and for $R > 0$, define*

$$PW_R = \big\{(h_{p-1}, h_p) \in PW_R(\mathbb{C})^2_{even} : h_{p-1}(\pm\lambda_{p-1}) = h_p(\pm\lambda_p)\big\}.$$

Then $\mathcal{H} : F \mapsto (\mathcal{H}^p_{p-1}(F), \mathcal{H}^p_p(F))$ is a topological linear isomorphism from $C_R^\infty(G, K, \tau_p, \tau_p)$ onto PW_R.

[1] In the sequel, 'topological linear isomorphism' will always mean 'bi-continuous isomorphism between vector spaces endowed with a Fréchet topology'.

Proof of Theorems 6.4 and 6.5:

• *First step :* using formulæ (6.3) and (6.4), we can see that each Fourier transform $\mathcal{H}_q^p$ maps continuously $C_R^\infty(G,K,\tau_p,\tau_p)$ into $PW_R(\mathbb{C})_{even}$. Indeed, the quantities between braces in (6.3) and (6.4) are exactly 'scalar expressions' of either dF or d^*F, and belong then to the space $C_R^\infty(\mathbb{R})_{odd}$. Thus, dividing by $\operatorname{sh} t$ is well-defined and yields $C_R^\infty(\mathbb{R})_{even}$ functions. According to the Paley-Wiener theorem for $\mathcal{J}^{(\frac{n}{2},-\frac{1}{2})}$, $\mathcal{H}_q^p(F) \in PW_R(\mathbb{C})_{even}$ and $\mathcal{H}_q^p$ is a continous map. The fact that $\mathcal{H}_{p-1}^p(F)(\pm\lambda_{p-1}) = \mathcal{H}_p^p(F)(\pm\lambda_p)$ is implied by the following result.

Lemma 6.6. *Let p be generic. Then $\Phi_{p-1}^p(\pm\lambda_{p-1},\cdot) \equiv \Phi_p^p(\pm\lambda_p,\cdot)$.*

Proof : this is due to the fact (Proposition 4.5) that these two functions verify the same differential equation $d\Phi = 0$ (or, equivalently, $d^*\Phi = 0$) with the same normalization. ✓

• *Second step :* conversely, let $h = (h_{p-1}, h_p) \in PW_R$. We put

$$F_h(x) = \sum_q \int_0^\infty d\nu_q(\lambda)\, h_q(\lambda)\Phi_q^p(\lambda, x). \tag{6.16}$$

(The map $h \mapsto F_h$ is the classical *wave packet transform.*) It is clear that F_h is τ_p-radial, and that its scalar components f_{p-1}, f_p live in $C^\infty(\mathbb{R})_{even}$ if h_{p-1}, h_p are entire. Let us prove the support preserving property for the scalar component f_{p-1} (the method is similar for f_p).
We have

$$f_{p-1}(t) = \frac{2}{\pi}\int_0^\infty \frac{d\lambda}{|c(\lambda)|^2}\Big\{\frac{np}{\lambda^2+[\rho-(p-1)]^2} h_{p-1}(\lambda)\tilde{\phi}_\lambda^{(\frac{n}{2},-\frac{1}{2})}(t) + \frac{n(n-p)}{\lambda^2+(\rho-p)^2} h_p(\lambda)\phi_\lambda^{(\frac{n}{2},-\frac{1}{2})}(t)\Big\},$$

where $\tilde{\phi}_\lambda^{(\frac{n}{2},-\frac{1}{2})} = \frac{n}{p}\phi_\lambda^{(\frac{n}{2}-1,-\frac{1}{2})} - \frac{n-p}{p}\phi_{\lambda,0,1}^{(\frac{n}{2},-\frac{3}{2})}$ (see Theorem 5.4). Now, we use the development (given in [Koo84], relation (2.17))

$$\phi_\lambda^{(\alpha,\beta)}(t) = c^{(\alpha,\beta)}(\lambda)\Phi_\lambda^{(\alpha,\beta)}(t) + c^{(\alpha,\beta)}(-\lambda)\Phi_{-\lambda}^{(\alpha,\beta)}(t),$$

where $\Phi_\lambda^{(\alpha,\beta)}(t)$ (defined for $\lambda \notin -i\,\mathbb{N}^*$) has a 'Harish-Chandra type' expansion:

$$\Phi_\lambda^{(\alpha,\beta)}(t) = e^{[i\lambda-(\alpha+\beta+1)]t}\sum_{m=0}^{+\infty}\Gamma_m(\lambda)e^{-mt}$$

(see *ibid.*, relation (6.6)). In our case, this gives rise to the developments

$$\phi_\lambda(t) = c(\lambda)\Phi_\lambda(t) + c(-\lambda)\Phi_{-\lambda}(t),$$
$$\tilde{\phi}_\lambda(t) = c(\lambda)\tilde{\Phi}_\lambda(t) + c(-\lambda)\tilde{\Phi}_{-\lambda}(t),$$

where we dropped the upper indices $(\alpha,\beta) = (\frac{n}{2}, -\frac{1}{2})$. Notice that the previous lemma implies that $\tilde{\phi}_{\lambda_{p-1}} = \phi_{\lambda_p}$ and, as a consequence, that $c(\lambda_{p-1})\tilde{\Phi}_{\lambda_{p-1}} = c(\lambda_p)\Phi_{\lambda_p}$. In other words,

$$\tilde{\Phi}_{\lambda_{p-1}} = \frac{c(\lambda_p)}{c(\lambda_{p-1})}\Phi_{\lambda_p} = -\frac{\rho-p+1}{p}\Phi_{\lambda_p}. \tag{6.17}$$

Now,

$$\begin{aligned} f_{p-1}(t) &= \frac{2}{\pi}\int_{-\infty}^{\infty}\frac{d\lambda}{c(-\lambda)}\Big\{\frac{np}{\lambda^2+[\rho-(p-1)]^2}h_{p-1}(\lambda)\tilde{\Phi}_\lambda(t) \\ &\qquad\qquad + \frac{n(n-p)}{\lambda^2+(\rho-p)^2}h_p(\lambda)\Phi_\lambda(t)\Big\} \\ &:= \int_{-\infty}^{\infty} d\lambda\, I(\lambda,t). \end{aligned}$$

Since $c(-\lambda)^{-1}\Phi_\lambda$ and $c(-\lambda)^{-1}\tilde{\Phi}_\lambda$ have only poles on the negative part of the imaginary axis, the integrand $I(\lambda,t)$ has two (simple) poles in the upper half-plane, at $\lambda=\lambda_p$ and $\lambda=\lambda_{p-1}$. In order to apply the residue theorem, we integrate the meromorphic function $\lambda\mapsto I(\lambda,t)$ on the square contour with vertices $r, r+ir, -r+ir, -r$, for a positive real number r. As soon as $r>\rho-p+1$, one gets

$$\int_{-r}^{r} d\lambda\, I(\lambda,t) = 2i\pi\big\{\mathrm{Res}_{\lambda=\lambda_p} I(\lambda,t) + \mathrm{Res}_{\lambda=\lambda_{p-1}} I(\lambda,t)\big\} + \int_{-r+ir}^{r+ir} d\lambda\, I(\lambda,t).$$

Using the Paley-Wiener type condition for h and the asymptotics estimates for the functions Φ_λ and $\tilde{\Phi}_\lambda$, one sees that $\int_{-r+ir}^{r+ir} d\lambda\, I(\lambda,t) \underset{r\to+\infty}{\longrightarrow} 0$ if $t>R$. Hence, since $h_{p-1}(\lambda_{p-1}) = h_p(\lambda_p)$, and using (6.17), one gets, for $t>R$,

$$\begin{aligned} f_{p-1}(t) &= \frac{4i\sqrt{\pi}}{2^n\Gamma(\frac{n}{2}+1)}\Big\{-\frac{\Gamma(\rho-p+1+\frac{n+1}{2})}{\Gamma(\rho-p+1)}\frac{np}{2i(\rho-p+1)}\frac{\rho-p+1}{p} \\ &\qquad\qquad + \frac{\Gamma(\rho-p+\frac{n+1}{2})}{\Gamma(\rho-p)}\frac{n(n-p)}{2i(\rho-p)}\Big\}h_p(\lambda_p)\,\Phi_{\lambda_p}(t) \\ &= \frac{\sqrt{\pi}}{2^{n-1}\Gamma(\frac{n}{2}+1)}\frac{\Gamma(n-p)}{\Gamma(\rho-p)}\Big\{-\frac{n-p}{\rho-p}\frac{np}{\rho-p+1}\frac{\rho-p+1}{p} \\ &\qquad\qquad + \frac{n(n-p)}{\rho-p}\Big\}h_p(\lambda_p)\,\Phi_{\lambda_p}(t) \\ &= 0. \end{aligned}$$

Thus, we have proved that if $h\in PW_R$, $F_h\in C_R^\infty(G,K,\tau_p,\tau_p)$.

• *Third step :* it remains to show that $\mathcal{H}(F_h) = h$ to conclude the proof of the Paley-Wiener theorem for $\mathcal{H}$ and to extend the validity of the inversion formula to the whole space $C_R^\infty(G, K, \tau_p, \tau_p)$. We begin with noticing that it suffices to show that

$$\big(\mathcal{H}(F_h), \mathcal{H}(G)\big) = \big(h, \mathcal{H}(G)\big)$$

for all $G \in L^2$ [2], and even for all $G \in V_p$ (by density). Then, using the Plancherel theorem above and the definition of F_h, we have:

$$\begin{aligned}
\big(\mathcal{H}(F_h), \mathcal{H}(G)\big) &= \big(F_h, G\big) \\
&= \int_G dx \ \mathrm{tr}\{F_h(x) G(x)^*\} \\
&= \sum_q \int_0^\infty d\nu_q(\lambda)\, h_q(\lambda) \int_G dx \ \mathrm{tr}\{\Phi_q^p(\lambda, x) G(x)^*\} \\
&= C_n^p \sum_q \int_0^\infty d\nu_q(\lambda)\, h_q(\lambda) \overline{\mathcal{H}_q^p(G)(\lambda)} \\
&= \big(h, \mathcal{H}(G)\big) \ .
\end{aligned}$$

Q.E.D. ✓

Remarks:

1. With $h = (h_{p-1}, h_p) \in PW_R$, associate F_h by (6.16). Then the scalar components f_{p-1}, f_p of F_h are also given by the following expressions:

$$\begin{aligned}
f_{p-1}(t) = {} & \mathcal{J}^{(\frac{n}{2}, -\frac{1}{2})^{-1}} \Big(\frac{4n(n-p)}{\lambda^2 + (\rho - p)^2}\, h_p \Big)(t) \\
& - (\operatorname{ch} t)\, \mathcal{J}^{(\frac{n}{2}, -\frac{1}{2})^{-1}} \Big(\frac{4n(n-p)}{\lambda^2 + [\rho - (p-1)]^2}\, h_{p-1} \Big)(t) \\
& + \mathcal{J}^{(\frac{n}{2}-1, -\frac{1}{2})^{-1}} \Big(\frac{\lambda^2 + (\frac{n-1}{2})^2}{\lambda^2 + [\rho - (p-1)]^2}\, h_{p-1} \Big)(t),
\end{aligned} \tag{6.18}$$

$$\begin{aligned}
f_p(t) = {} & \mathcal{J}^{(\frac{n}{2}, -\frac{1}{2})^{-1}} \Big(\frac{4np}{\lambda^2 + [\rho - (p-1)]^2}\, h_{p-1} \Big)(t) \\
& - (\operatorname{ch} t)\, \mathcal{J}^{(\frac{n}{2}, -\frac{1}{2})^{-1}} \Big(\frac{4np}{\lambda^2 + (\rho - p)^2}\, h_p \Big)(t) \\
& + \mathcal{J}^{(\frac{n}{2}-1, -\frac{1}{2})^{-1}} \Big(\frac{\lambda^2 + (\frac{n-1}{2})^2}{\lambda^2 + (\rho - p)^2}\, h_p \Big)(t).
\end{aligned} \tag{6.19}$$

[2] The scalar products are taken in the space $L^2(\mathbb{R}; C_n^p d\nu_{p-1}, C_n^p d\nu_p)_{even}$.

2. Theorem 6.5 is a special case of Campoli's Paley-Wiener theorem for real rank one semisimple Lie groups: see [Cam80], §§2.3.1, 2.3.2 and especially 3.2.1. In particular, Campoli described the image of a Fourier transform defined on the algebra of K-central functions $f \in C_c^\infty(G)$ verifying the K-finiteness condition $f = \overline{\chi_\tau} * f * \chi_\tau$, where $\tau \in \widehat{K}$ and $\chi_\tau(k) := (\dim \tau) \operatorname{tr} \tau(k)$, and this algebra is isomorphic to $C_c^\infty(G, K, \tau, \tau)$ (see our remark following Theorem 5.1). Recall also that Campoli's results have been extended to higher rank real reductive groups by Arthur ([Art83], Theorems 3.3 and 4.1), and the generalization to the case of reductive symmetric spaces G/H has been obtained recently by van den Ban and Schlichtkrull (see the announcement in [BFJS97]).

An alternative proof of Theorems 6.3, 6.4 and 6.5 can be given, by working out spherical analysis in Schwartz spaces. This approach has the advantage to avoid the use of the Hodge-Kodaira decomposition for $L^2(G, K, \tau_p, \tau_p)$ — then of the subspace V_p we have introduced. We believe that this second technique is worthwhile, since it could be the unique way (or, at least, the easiest) to extend our results to more general homogeneous vector bundles over G/K (where the Hodge-Kodaira decomposition has no analogue).

Following Harish-Chandra, let us define the (L^2) Schwartz space for τ_p-radial functions on G:

$$\mathcal{S}(G, K, \tau_p, \tau_p) = \big\{ F \in C^\infty(G, K, \tau, \tau) \,:\, \forall D_1, D_2 \in U(\mathfrak{g}),\ \forall N \in \mathbb{N}, \\ \sup_{t \geqslant 0} \|F(D_1 : a_t : D_2)\|_{\operatorname{End} \mathcal{H}_{\tau_p}} (1+t)^N e^{\rho\, t} < +\infty \big\}.$$

Proceeding as in [GV88], §6.1, we can endow $\mathcal{S}(G, K, \tau_p, \tau_p)$ with a Fréchet topology, so that

$$\overline{C_c^\infty(G, K, \tau_p, \tau_p)} = \mathcal{S}(G, K, \tau_p, \tau_p), \tag{6.20}$$

$$\text{and} \quad \overline{\mathcal{S}(G, K, \tau_p, \tau_p)} = L^2(G, K, \tau_p, \tau_p). \tag{6.21}$$

Note that it is equivalent to say that $F \in \mathcal{S}(G, K, \tau_p, \tau_p)$ or that its scalar components live in the space $(\operatorname{ch} t)^{-\rho}\, \mathcal{S}(\mathbb{R})_{even}$, where $\mathcal{S}(\mathbb{R})$ is the classical Schwartz space on $\mathbb{R}$.

Theorem 6.7. *Let p be generic. The spherical Fourier transform $\mathcal{H}$ of τ_p-radial functions is a topological linear isomorphism between*

$$\mathcal{S}(G, K, \tau_p, \tau_p) \simeq (\operatorname{ch} t)^{-\rho}\, \mathcal{S}(\mathbb{R})_{even} \oplus (\operatorname{ch} t)^{-\rho}\, \mathcal{S}(\mathbb{R})_{even}$$

and $\mathcal{S}(\mathbb{R})_{even} \oplus \mathcal{S}(\mathbb{R})_{even}$.

Proof : we remark first that Proposition 6.1 is still valid in the Schwartz setting. $\mathcal{H}_q^p(F)$ is then the Jacobi transform of an expression which lives in the space $(\operatorname{ch} t)^{-\rho-1}\,\mathcal{S}(\mathbb{R})_{even}$. Now, let us recall that there exists an *Abel transform* $\mathcal{A}^{(\alpha,\beta)}$ such that

$$\mathcal{J}^{(\alpha,\beta)} = \mathcal{F} \circ \mathcal{A}^{(\alpha,\beta)},$$

where $\mathcal{F}$ denotes the usual Euclidean Fourier transform (the reader will find the definitions of $\mathcal{A}^{(\alpha,\beta)}$ and $\mathcal{F}$ in §7.1). According to [Koo84], §6, we have the topological isomorphism

$$\mathcal{A}^{(\alpha,\beta)} : (\operatorname{ch} t)^{-\sigma}\,\mathcal{S}(\mathbb{R})_{even} \xrightarrow{\simeq} (\operatorname{ch} t)^{-\sigma+\alpha+\beta+1}\,\mathcal{S}(\mathbb{R})_{even}$$

for $\alpha \geqslant \beta \geqslant -\frac{1}{2}$ and $\sigma \geqslant \alpha+\beta+1$. Hence

$$\mathcal{J}^{(\frac{n}{2},-\frac{1}{2})} : (\operatorname{ch} t)^{-\rho-1}\,\mathcal{S}(\mathbb{R})_{even} \underset{\simeq}{\xrightarrow{\mathcal{A}^{(\frac{n}{2},-\frac{1}{2})}}} \mathcal{S}(\mathbb{R})_{even} \underset{\simeq}{\xrightarrow{\mathcal{F}}} \mathcal{S}(\mathbb{R})_{even}, \tag{6.22}$$

so that $\mathcal{H}$ maps continuously $\mathcal{S}(G,K,\tau_p,\tau_p)$ into $\mathcal{S}(\mathbb{R})_{even} \oplus \mathcal{S}(\mathbb{R})_{even}$. The injectivity of $\mathcal{H}$ is obvious from Proposition 6.1, and it remains to show the surjectivity.
Let $(h_{p-1},h_p) \in \mathcal{S}(\mathbb{R})_{even} \oplus \mathcal{S}(\mathbb{R})_{even}$ and put

$$g_q = -4n\,\mathcal{J}^{(\frac{n}{2},-\frac{1}{2})^{-1}}(h_q) \quad \in (\operatorname{ch} t)^{-\rho-1}\,\mathcal{S}(\mathbb{R})_{even}.$$

We have to solve the following system:

$$(S) \quad \begin{cases} (\operatorname{sh} t) g_p = f_p' + p(\coth t) f_p - \frac{p}{\operatorname{sh} t} f_{p-1}, \\ (\operatorname{sh} t) g_{p-1} = f_{p-1}' + (n-p)(\coth t) f_{p-1} - \frac{n-p}{\operatorname{sh} t} f_p. \end{cases}$$

Since h_{p-1} and h_p are independent, we can suppose first $h_p \equiv 0$, i.e. $g_p \equiv 0$. Then, as in the proof of Proposition 6.1 (and with the same notation), (S) gives

$$\{L + p(n-p+1)\} f_p = p\, g_{p-1},$$

hence

$$\mathcal{J}^{(\frac{n}{2},-\frac{1}{2})}(f_p)(\lambda) = \frac{4np}{\lambda^2 + (\rho-p+1)^2} h_{p-1}.$$

Thus, if we put

$$f_p = \mathcal{J}^{(\frac{n}{2},-\frac{1}{2})^{-1}}\Big(\frac{4np}{\lambda^2+(\rho-p+1)^2} h_{p-1}\Big) \quad \in (\operatorname{ch} t)^{-\rho-1}\,\mathcal{S}(\mathbb{R})_{even} \quad \text{(by (6.22))},$$

and since

$$f_{p-1} = \frac{\operatorname{sh} t}{p} f_p' + (\operatorname{ch} t) f_p \quad \in (\operatorname{ch} t)^{-\rho}\, \mathcal{S}(\mathbb{R})_{even},$$

we have $F \in \mathcal{S}(G, K, \tau_p, \tau_p)$, verifying $\mathcal{H}^p_{p-1}(F) = h_{p-1}$, and furthermore $dF = 0$, $\mathcal{H}^p_p(F) = 0$ (by Proposition 6.1).

Similarly, if we assume now $h_{p-1} \equiv 0$, i.e. $g_{p-1} \equiv 0$, we can take as solutions of (S):

$$f_{p-1} = \mathcal{J}^{(\frac{n}{2},-\frac{1}{2})^{-1}} \Big(\frac{4n(n-p)}{\lambda^2 + (\rho - p)^2} h_p \Big) \quad \in (\operatorname{ch} t)^{-\rho-1}\, \mathcal{S}(\mathbb{R})_{even},$$

$$\text{and} \quad f_p = \frac{\operatorname{sh} t}{n-p} f_{p-1}' + (\operatorname{ch} t) f_{p-1} \quad \in (\operatorname{ch} t)^{-\rho}\, \mathcal{S}(\mathbb{R})_{even},$$

so that $F \in \mathcal{S}(G, K, \tau_p, \tau_p)$, verifying $\mathcal{H}^p_p(F) = h_p$, $d^* F = 0$, and $\mathcal{H}^p_{p-1}(F) = 0$.

Finally, we obtain the general solution of (S) by superposing the expressions of the two particular cases we have considered. ✓

Corollary 6.8. *The previous result implies the following statements:*

(i) *the inversion formula (6.14) and the Plancherel formula (6.15) for $F \in \mathcal{S}(G, K, \tau_p, \tau_p)$, and, by restriction, for $F \in C_c^\infty(G, K, \tau_p, \tau_p)$;*

(ii) *Theorem 6.3 (ii) (Plancherel Theorem);*

(iii) *Theorem 6.5 (Paley-Wiener Theorem).*

Proof :

(i) is easy.

(ii) follows from (6.21).

(iii) is given by the first and second steps in the proof of Theorem 6.5, since (i) is known. ✓

Special case $p = \frac{n-1}{2}$

Let $F \in C_c^\infty(G, K, \tau_p, \tau_p)$. Analogously to (6.1), its *(spherical) Fourier transform* $\mathcal{H}(F)$ is the triple $(\mathcal{H}^p_{p-1}(F), \mathcal{H}^p_{p,+}(F), \mathcal{H}^p_{p,-}(F))$ of functions defined on $\mathbb{C}$ by:

$$\mathcal{H}^p_{p-1}(F)(\lambda) = \frac{1}{C_n^p} \big(F, \Phi^p_{p-1}(\lambda, \cdot^{-1})^* \big), \tag{6.23}$$

$$\mathcal{H}^p_{p,\pm}(F)(\lambda) = \frac{1}{C_n^p} \big(F, \Phi^p_{p,\pm}(\lambda, \cdot^{-1})^* \big)\,. \tag{6.24}$$

We shall first make these expressions more explicit.

Proposition 6.9. *Let* $p = \frac{n-1}{2}$ *and let* $F \in C_c^\infty(G, K, \tau_p, \tau_p)$ *with scalar components* f_{p-1}, f_p^+, f_p^-. *Recall that we have put* $f_p = \frac{1}{2}(f_p^+ + f_p^-)$ *and* $\tilde{f}_p = \frac{1}{2}(f_p^+ - f_p^-)$. *Then:*

(i) *the spherical Fourier transforms can be expressed in terms of Jacobi transforms:*

$$\mathcal{H}^p_{p-1}(F)(\lambda) = -\frac{1}{4n}\mathcal{J}^{(\frac{n}{2},-\frac{1}{2})}\Big(\frac{1}{\operatorname{sh} t}\{f'_{p-1}(t) + \tfrac{n+1}{2}(\coth t) f_{p-1}(t) - \tfrac{n+1}{2}(\operatorname{sh} t)^{-1} f_p(t)\}\Big)(\lambda), \tag{6.25}$$

$$\mathcal{H}^p_{p,\pm}(F)(\lambda) = -\frac{1}{4n}\mathcal{J}^{(\frac{n}{2},-\frac{1}{2})}\Big(\frac{1}{\operatorname{sh} t}\{f'_{p}(t) + \tfrac{n-1}{2}(\coth t) f_{p}(t) - \tfrac{n-1}{2}(\operatorname{sh} t)^{-1} f_{p-1}(t)\}\Big)(\lambda) \pm \frac{i\lambda}{4n}\mathcal{J}^{(\frac{n}{2},-\frac{1}{2})}\Big(\frac{\tilde{f}_p}{\operatorname{sh} t}\Big)(\lambda). \tag{6.26}$$

(ii) $\mathcal{H}^p_{p,\pm}(F) \equiv 0$ *if and only if* $dF = 0$, *and in that case*

$$\mathcal{H}^p_{p-1}(F)(\lambda) = \frac{\lambda^2+1}{2n(n-1)}\mathcal{J}^{(\frac{n}{2},-\frac{1}{2})}(f_p^\pm)(\lambda); \tag{6.27}$$

$\mathcal{H}^p_{p-1}(F) \equiv 0$ *if and only if* $d^*F = 0$, *and in that case*

$$\mathcal{H}^p_{p,\pm}(F)(\lambda) = \frac{\lambda^2}{2n(n+1)}\mathcal{J}^{(\frac{n}{2},-\frac{1}{2})}(f_{p-1})(\lambda) \pm \frac{i\lambda}{4n}\mathcal{J}^{(\frac{n}{2},-\frac{1}{2})}\Big(\frac{\tilde{f}_p}{\operatorname{sh} t}\Big)(\lambda). \tag{6.28}$$

Proof : we globally proceed as in the generic case (Proposition 6.1). In particular, the proof of (6.27) is identical to the one of (6.4), since the scalar components $\varphi_p^\pm(\lambda,\cdot)$ of $\Phi^p_{p-1}(\lambda,\cdot)$ are equal.
Let us deal now with (6.26). Recall from Section 5 that the scalar components of $\Phi^p_{p,\pm}(\lambda,\cdot)$ are given by

$$\begin{aligned}
\varphi_{p-1}(\lambda,t) &= \phi_\lambda^{(\frac{n}{2},-\frac{1}{2})}(t),\\
\varphi_p(\lambda,t) &= \tfrac{2\operatorname{sh} t}{n+1}\varphi'_{p-1}(\lambda,t) + (\operatorname{ch} t)\varphi_{p-1}(\lambda,t),\\
\varphi_p^+(\lambda,t) &= \varphi_p(\lambda,t) \mp \tfrac{2i\lambda}{n+1}(\operatorname{sh} t)\varphi_{p-1}(\lambda,t),\\
\varphi_p^-(\lambda,t) &= \varphi_p(\lambda,t) \pm \tfrac{2i\lambda}{n+1}(\operatorname{sh} t)\varphi_{p-1}(\lambda,t).
\end{aligned}$$

Let $I_\pm(\lambda) = C_n^p\,\mathcal{H}^p_{p,\pm}(F)(\lambda) = \big(F, \Phi^p_{p,\pm}(\lambda,\cdot^{-1})^*\big)$. Using (5.5), and since

$$\varphi_{p-1}(\lambda,\cdot) \quad \text{is even,} \quad \varphi_p^\pm(\lambda,-t) = \varphi_p^\mp(\lambda,t),$$

we have

$$I_\pm(\lambda) = \int_0^\infty dt\,(2\operatorname{sh} t)^{n-1}\{C_{n-1}^{p-1}\, f_{p-1}(t)\varphi_{p-1}(\lambda,t) + \tfrac{1}{2}C_{n-1}^{p}\, f_p^+(t)\varphi_p^-(\lambda,t) + \tfrac{1}{2}C_{n-1}^{p}\, f_p^-(t)\varphi_p^+(\lambda,t).$$

In other terms, $I_\pm(\lambda) = J_{p-1}(\lambda) + J_{p,\pm}^+(\lambda) + J_{p,\pm}^-(\lambda)$, where

$$\begin{aligned}
J_{p-1}(\lambda) &= C_{n-1}^{p-1}\int_0^\infty dt\,(2\operatorname{sh} t)^{n-1} f_{p-1}(t)\varphi_{p-1}(\lambda,t),\\
J_{p,\pm}^+(\lambda) &= \tfrac{1}{2}C_{n-1}^{p}\int_0^\infty dt\,(2\operatorname{sh} t)^{n-1} f_p^+(t)\Big[\tfrac{2}{n+1}(\operatorname{sh} t)\varphi_{p-1}'(\lambda,t)\\
&\qquad + (\operatorname{ch} t)\varphi_{p-1}(\lambda,t) \pm \tfrac{2i\lambda}{n+1}(\operatorname{sh} t)\varphi_{p-1}(\lambda,t)\Big],\\
J_{p,\pm}^-(\lambda) &= \tfrac{1}{2}C_{n-1}^{p}\int_0^\infty dt\,(2\operatorname{sh} t)^{n-1} f_p^-(t)\Big[\tfrac{2}{n+1}(\operatorname{sh} t)\varphi_{p-1}'(\lambda,t)\\
&\qquad + (\operatorname{ch} t)\varphi_{p-1}(\lambda,t) \mp \tfrac{2i\lambda}{n+1}(\operatorname{sh} t)\varphi_{p-1}(\lambda,t)\Big].
\end{aligned}$$

Then

$$\begin{aligned}
&J_{p,\pm}^+(\lambda) + J_{p,\pm}^-(\lambda)\\
&\quad = \tfrac{1}{2}C_{n-1}^{p}\int_0^\infty dt\,(2\operatorname{sh} t)^{n-1}[f_p^+(t) + f_p^-(t)]\Big[\tfrac{2}{n+1}(\operatorname{sh} t)\varphi_{p-1}'(\lambda,t)\\
&\qquad\qquad + (\operatorname{ch} t)\varphi_{p-1}(\lambda,t)\Big]\\
&\quad \pm C_{n-1}^{p}\,\tfrac{i\lambda}{n+1}\int_0^\infty dt\,(2\operatorname{sh} t)^{n-1}[f_p^+(t) - f_p^-(t)](\operatorname{sh} t)\varphi_{p-1}(\lambda,t).
\end{aligned}$$

Since $f_p^+ + f_p^- = 2f_p$, we can see, as in the proof of Proposition 6.1, that the first integral in the last expression, added to $J_{p-1}(\lambda)$, gives the quantity

$$-\tfrac{C_{n-1}^{p}}{4}\mathcal{J}^{(\frac{n}{2},-\frac{1}{2})}\Big(\tfrac{1}{\operatorname{sh} t}\big\{\tfrac{2}{n-1}f_p'(t) + (\coth t)f_p(t) - (\operatorname{sh} t)^{-1}f_{p-1}(t)\big\}\Big),$$

while the second integral is

$$\pm C_{n-1}^{p}\,\tfrac{i\lambda}{2(n+1)}\mathcal{J}^{(\frac{n}{2},-\frac{1}{2})}\big((\operatorname{sh} t)^{-1}\tilde{f}_p(t)\big)(\lambda).$$

(Note that the quotient $(\operatorname{sh} t)^{-1}\tilde{f}_p(t)$ is well-defined at $t=0$ since the two functions are odd.) Putting these calculations together, we get the expression of $I_\pm(\lambda)$, hence (6.26).

The rest of the proposition is left to the reader, since the arguments are identical to the ones in the proof of Proposition 6.1. ✓

In order to state the main results of this subsection (Paley-Wiener and Plancherel theorems, inversion formula), we proceed as in the generic case.

First, we establish a intermediate inversion formula for the spherical transform, that is to say when F is in the subspace V_p defined by (6.8). This is done by inverting the expressions (6.27) and (6.28) thanks to the inversion formula for the Jacobi transform:

$$f_p(t) = \frac{1}{2\pi}\int_0^\infty \frac{2n(n-1)}{\lambda^2+1}\frac{d\lambda}{|c(\lambda)|^2}\mathcal{H}^p_{p-1}(F)(\lambda)\phi_\lambda^{(\frac{n}{2},-\frac{1}{2})}(t), \tag{6.29}$$

$$\begin{aligned} f_{p-1}(t) = \frac{1}{2\pi}\int_0^\infty \frac{2n(n+1)}{\lambda^2}\frac{d\lambda}{|c(\lambda)|^2}\frac{1}{2}\{&\mathcal{H}^p_{p,+}(F)(\lambda) \\ &+\mathcal{H}^p_{p,-}(F)(\lambda)\}\phi_\lambda^{(\frac{n}{2},-\frac{1}{2})}(t), \end{aligned} \tag{6.30}$$

$$\begin{aligned} \tilde{f}_p(t) = \frac{1}{2\pi}\operatorname{sh} t\int_0^\infty \frac{4n}{i\lambda}\frac{d\lambda}{|c(\lambda)|^2}\frac{1}{2}\{&\mathcal{H}^p_{p,+}(F)(\lambda) \\ &-\mathcal{H}^p_{p,-}(F)(\lambda)\}\phi_\lambda^{(\frac{n}{2},-\frac{1}{2})}(t), \end{aligned} \tag{6.31}$$

where Harish-Chandra's c-function was defined in (6.11). These formulæ suffice to express completely F, since $f_p^\pm = f_p \pm \tilde{f}_p$.

We define now the Plancherel measures as in (6.12) and (6.13):

$$d\nu_{p-1}(\lambda) = \frac{n(n-1)}{\pi(\lambda^2+1)}\frac{d\lambda}{|c(\lambda)|^2}, \qquad d\nu_p(\lambda) = \frac{n(n+1)}{\pi\lambda^2}\frac{d\lambda}{|c(\lambda)|^2}.$$

Since the method is identical to the one given in the generic case, we give now the main results, without proofs (except for the Paley-Wiener theorem). We keep the same notations.

Theorem 6.10 (Inversion formula). *Let $p=\frac{n-1}{2}$. The spherical Fourier transform $\mathcal{H}=(\mathcal{H}^p_{p-1},\mathcal{H}^p_{p,+},\mathcal{H}^p_{p,-})$ of τ_p-radial functions $F\in C_c^\infty(G,K,\tau_p,\tau_p)$ is inverted by the following formula:*

$$\begin{aligned} F(x) = \int_0^\infty \big\{&d\nu_{p-1}(\lambda)\mathcal{H}^p_{p-1}(F)(\lambda)\Phi^p_{p-1}(\lambda,x) \\ &+\frac{1}{2}d\nu_p(\lambda)[\mathcal{H}^p_{p,+}(F)(\lambda)\Phi^p_{p,+}(\lambda,x)+\mathcal{H}^p_{p,-}(F)(\lambda)\Phi^p_{p,-}(\lambda,x)]\big\}. \end{aligned}$$

Theorem 6.11 (Plancherel). *Let* $p = \frac{n-1}{2}$. *Define the space*

$$\begin{aligned} L^2(\mathbb{R}; \mu_1, \mu_2, \mu_2) = &\{(h_1, h_2^+, h_2^-) : h_1 \text{ is even}, \ h_2^\pm(-\lambda) = h_2^\mp(\lambda), \\ &\qquad h_i^{(\pm)} \in L^2(\mathbb{R}, \mu_i), \ i = 1, 2\}. \end{aligned}$$

Then:

(i) *Plancherel formula: for* $F \in V_p$,

$$\begin{aligned} \|F\|_{L^2}^2 &= \int_G dx\, \|F(x)\|_{HS}^2 \\ &= \int_0^\infty dt\, (2\operatorname{sh} t)^{n-1} \big\{ C_{n-1}^{p-1} |f_{p-1}(t)|^2 \\ &\qquad + \tfrac{1}{2} C_{n-1}^p |f_p^+(t)|^2 + \tfrac{1}{2} C_{n-1}^p |f_p^-(t)|^2 \big\} \\ &= C_n^p \int_0^\infty \big\{ d\nu_{p-1}(\lambda)\, |\mathcal{H}_{p-1}^p(F)(\lambda)|^2 \\ &\qquad + \tfrac{1}{2} d\nu_p(\lambda) (|\mathcal{H}_{p,+}^p(F)(\lambda)|^2 + \mathcal{H}_{p,-}^p(F)(\lambda)|^2) \big\} \\ &= C_n^p \int_0^\infty \big\{ d\nu_{p-1}(\lambda)\, |\mathcal{H}_{p-1}^p(F)(\lambda)|^2 + d\nu_p(\lambda)\, |\mathcal{H}_{p,\pm}^p(F)(\lambda)|^2 \big\}. \end{aligned}$$

(ii) *The spherical Fourier transform* $\mathcal{H}$ *of* τ_p*-radial functions extends to a bijective isometry from*

$$\begin{aligned} &L^2(G, K, \tau_p, \tau_p) \\ &\qquad \simeq L^2(\mathbb{R}; C_{n-1}^{p-1} (2\operatorname{sh} t)^{n-1} dt, \tfrac{C_{n-1}^p}{2} (2\operatorname{sh} t)^{n-1} dt, \tfrac{C_{n-1}^p}{2} (2\operatorname{sh} t)^{n-1} dt) \end{aligned}$$

onto $L^2(\mathbb{R}; C_n^p\, d\nu_{p-1}, \frac{1}{2} C_n^p\, d\nu_p, \frac{1}{2} C_n^p\, d\nu_p)$.

Theorem 6.12 (Paley-Wiener). *Let* $p = \frac{n-1}{2}$. *For* $R > 0$, *define*

$$\begin{aligned} PW_R = \big\{ &(h_{p-1}, h_{p,+}, h_{p,-}) \in PW_R(\mathbb{C})_{even} \times PW_R(\mathbb{C}) \times PW_R(\mathbb{C}) : \\ &\qquad h_{p,\pm}(\lambda) = h_{p,\mp}(-\lambda), \ h_{p-1}(\pm i) = h_{p,\pm}(0) \big\}. \end{aligned}$$

Then $\mathcal{H} : F \mapsto (\mathcal{H}_{p-1}^p(F), \mathcal{H}_{p,+}^p(F), \mathcal{H}_{p,-}^p(F))$ *is a topological linear isomorphism between* $C_R^\infty(G, K, \tau_p, \tau_p)$ *and* PW_R.

Proof of Theorem 6.12 (sketch): it is similar to the one given in the generic case, except for the details of the step consisting in showing the implication $h \in PW_R \Rightarrow F_h \in C_R^\infty(G, K, \tau_p, \tau_p)$.

The wave packet of a function $h \in PW_R$ is here expressed as:

$$F_h(x) = \int_0^\infty \Big\{ d\nu_{p-1}(\lambda)\, h_{p-1}(\lambda)\Phi^p_{p-1}(\lambda, x) + \frac{1}{2} d\nu_p(\lambda)[h_{p,+}(\lambda)\Phi^p_{p,+}(\lambda, x) + h_{p,-}(\lambda)\Phi^p_{p,-}(\lambda, x)] \Big\}.$$

We want to prove that $f_{p-1}(t) = 0$ as soon as $t > R$ (the same argument works for $f_p^{\pm}$). Proceeding as in the proof of Theorem 6.5 (and with the same notations), we have

$$\begin{aligned} f_{p-1}(t) &= \frac{1}{\pi} \int_0^\infty \frac{d\lambda}{|c(\lambda)|^2} \Big\{ \frac{n(n-1)}{\lambda^2+1} h_{p-1}(\lambda)\tilde{\phi}_\lambda(t) \\ &\qquad + \frac{1}{2}\frac{n(n+1)}{\lambda^2}[h_{p,+}(\lambda) + h_{p,-}(\lambda)]\phi_\lambda(t) \Big\} \\ &= \frac{1}{\pi} \int_{-\infty}^\infty \frac{d\lambda}{c(-\lambda)} \Big\{ \frac{n(n-1)}{\lambda^2+1} h_{p-1}(\lambda)\tilde{\Phi}_\lambda(t) \\ &\qquad + \frac{1}{2}\frac{n(n+1)}{\lambda^2}[h_{p,+}(\lambda) + h_{p,-}(\lambda)]\Phi_\lambda(t) \Big\}. \end{aligned}$$

The pathological point is the following: when moving the integration contour (as in Theorem 6.5), we get two poles for the integrand, at $\lambda = 0$ and $\lambda = i$. But, while the second one is only due to the contribution of the quantity $(\lambda^2+1)^{-1}$, the first arises because of the product $[c(-\lambda)\lambda^2]^{-1}$. Indeed,

$$c(-\lambda)^{-1} = \frac{\sqrt{\pi}}{2^n\Gamma(\frac{n}{2}+1)} \prod_{k=0}^{(n-1)/2} (-i\lambda + k),$$

and the function $\lambda \mapsto [c(-\lambda)\lambda^2]^{-1}$ has a simple pole at $\lambda = 0$. Hence, remarking that $\tilde{\Phi}_i = -\frac{2}{n-1}\Phi_0$, when $t > R$ we get

$$\begin{aligned} f_{p-1}(t) &= 2i\pi \Big\{ \frac{-2n}{c(-i)} \operatorname{Res}_{\lambda=i} \frac{1}{\lambda^2+1} + \frac{n(n+1)}{2} \operatorname{Res}_{\lambda=0}[c(-\lambda)\lambda^2]^{-1} \Big\} h_{p,\pm}(0)\Phi_0(t) \\ &= \frac{\sqrt{\pi}}{2^n\Gamma(\frac{n}{2}+1)} (\tfrac{n-1}{2})!\, \{-n(n+1) + n(n+1)\} h_{p,\pm}(0)\Phi_0(t) \\ &= 0. \end{aligned}$$

✓

As in the generic case, Theorems 6.10, 6.11 and 6.12 can be derived from analysis in the Schwartz spaces setting. More precisely, we can prove the following result.

Theorem 6.13. *Let $p = \frac{n-1}{2}$. The spherical Fourier transform $\mathcal{H}$ is a topological linear isomorphism between*

$$\mathcal{S}(G,K,\tau_p,\tau_p) \simeq \{(f_{p-1}, f_p^+, f_p^-) \in (\operatorname{ch} t)^{-\rho}\mathcal{S}(\mathbb{R})_{even} \oplus (\operatorname{ch} t)^{-\rho}\mathcal{S}(\mathbb{R}) \oplus (\operatorname{ch} t)^{-\rho}\mathcal{S}(\mathbb{R}) : f_p^{\pm}(-t) = f_p^{\mp}(t)\}$$

and the space

$$\{(h_1, h_2^+, h_2^-) \in \mathcal{S}(\mathbb{R})_{even} \oplus \mathcal{S}(\mathbb{R}) \oplus \mathcal{S}(\mathbb{R}) \ : h_2^{\pm}(-\lambda) = h_2^{\mp}(\lambda)\}.$$

Special case $p = \frac{n}{2}$

For $F^{\pm} \in C_c^{\infty}(G,K,\tau^{\pm}_{\frac{n}{2}},\tau^{\pm}_{\frac{n}{2}})$, we put, similarly to (6.1),

$$\mathcal{H}^{\pm}(F^{\pm})(\lambda) = \frac{2}{C_n^{n/2}} \left(F^{\pm}, \Phi^{\pm}(\lambda, \cdot^{-1})^*\right). \tag{6.32}$$

Since one can restrict to A, (6.32) becomes:

$$\mathcal{H}^{\pm}(F^{\pm})(\lambda) = \int_0^{\infty} dt\, (2\operatorname{sh} t)^{n-1} f^{\pm}(t)\varphi^{\pm}(\lambda,t), \tag{6.33}$$

which shows that $\mathcal{H}^+$ and $\mathcal{H}^-$ have the same analytic expression, since $\varphi^+ = \varphi^-$. Let us now rewrite (6.33):

$$\begin{aligned}\mathcal{H}^{\pm}(F^{\pm})(\lambda) &= \int_0^{\infty} dt\, (2\operatorname{sh} t)^{n-1}(\operatorname{ch}\tfrac{t}{2})^2 f^{\pm}(t)\phi_{2\lambda}^{(\frac{n}{2}-1,\frac{n}{2}+1)}(\tfrac{t}{2})\\ &= \frac{1}{8}\int_0^{\infty} d(\tfrac{t}{2})\,(2\operatorname{sh}\tfrac{t}{2})^{n-1}(2\operatorname{ch}\tfrac{t}{2})^{n+3}\,\frac{f^{\pm}(t)}{(\operatorname{ch}\frac{t}{2})^2}\,\phi_{2\lambda}^{(\frac{n}{2}-1,\frac{n}{2}+1)}(\tfrac{t}{2})\\ &= \frac{1}{8}\mathcal{J}^{(\frac{n}{2}-1,\frac{n}{2}+1)}\Big(\frac{f^{\pm}(2\cdot)}{\operatorname{ch}^2}\Big)(2\lambda).\end{aligned}$$

Thus, the spherical Fourier transform is exactly a Jacobi transform (the division by the factor $(\operatorname{ch}\frac{t}{2})^2$ does not introduce any singularity, nor change in the size of the support of $f^{\pm}$). This is why we obtain immediately the following theorem.

Theorem 6.14 (Paley-Wiener). *Let $p = \frac{n}{2}$. The spherical Fourier transform $\mathcal{H}^{\pm}$ is a topological linear isomorphism from $C_R^{\infty}(G,K,\tau_p^{\pm},\tau_p^{\pm}) \simeq C_R^{\infty}(\mathbb{R})_{even}$ onto $PW_R(\mathbb{C})_{even}$.*

Remark: contrary to Theorems 6.5 and 6.12, in the definition of the Paley-Wiener space there are no conditions of equality between values of the functions at certain points. As was shown in [Cam80], Theorem 3.3.1, this simplification occurs for any rank one group G, as soon as the representation $\tau_{|_M}$ is irreducible.

To state the inversion formula for the scalar component $f^\pm$, it suffices again to use Theorem 2.3 of [Koo84]. But, this time, a discrete term appears in the formula:

$$f^\pm(t) = 8\,\mathrm{ch}^2\,\tfrac{t}{2}\Big\{\frac{1}{2\pi}\int_0^\infty \frac{d(2\lambda)}{|c(2\lambda)|^2}\mathcal{H}^\pm(F^\pm)(\lambda)\phi_{2\lambda}^{(\frac{n}{2}-1,\frac{n}{2}+1)}(\tfrac{t}{2}) + d(i)\mathcal{H}^\pm(F^\pm)(\tfrac{i}{2})\phi_i^{(\frac{n}{2}-1,\frac{n}{2}+1)}(\tfrac{t}{2})\Big\}, \tag{6.34}$$

where, with the notations of [Koo84], $d(i) = -i\,\mathrm{Res}_{2\lambda=i}[c(2\lambda)c(-2\lambda)]^{-1}$. In terms of representation theory, this discrete part comes from the contribution of the two discrete series discussed in Section 3 and Appendix A. From an analytic point of view, the discrete part is due to the existence of a pole near the origin ($\lambda = i/2$) for the function $\lambda \mapsto [c(2\lambda)c(-2\lambda)]^{-1}$ (i.e. for the Plancherel measure); let us see why.

Harish-Chandra's c-function is here given by:

$$c(\lambda) = c^{(\frac{n}{2}-1,\frac{n}{2}+1)}(\lambda) = 2^{n+1-i\lambda}\frac{\Gamma(\frac{n}{2})\Gamma(i\lambda)}{\Gamma(\frac{i\lambda+n+1}{2})\Gamma(\frac{i\lambda-1}{2})}.$$

(See [Koo84], formula (2.18).) Hence

$$[c(\lambda)c(-\lambda)]^{-1} = 2^{-2n-2}[(\tfrac{n}{2}-1)!]^{-2}\,\frac{\Gamma(\frac{i\lambda}{2}+\frac{n+1}{2})\Gamma(-\frac{i\lambda}{2}+\frac{n+1}{2})\Gamma(\frac{i\lambda}{2}-\frac{1}{2})\Gamma(-\frac{i\lambda}{2}-\frac{1}{2})}{\Gamma(i\lambda)\Gamma(-i\lambda)}.$$

Using the 'complements formula', as well as the usual properties of the Γ function, one has:

$$\begin{aligned}\Gamma(\tfrac{i\lambda}{2}-\tfrac{1}{2})\Gamma(-\tfrac{i\lambda}{2}-\tfrac{1}{2}) &= \frac{\Gamma(\frac{i\lambda}{2}+\frac{1}{2})}{\frac{i\lambda}{2}-\frac{1}{2}}\,\frac{\Gamma(-\frac{i\lambda}{2}+\frac{1}{2})}{-\frac{i\lambda}{2}-\frac{1}{2}}\\ &= \frac{\pi}{\mathrm{ch}\,\frac{\pi\lambda}{2}}\,\frac{1}{(\frac{\lambda}{2})^2+(\frac{1}{2})^2},\end{aligned}$$

and

$$\begin{aligned}&\frac{\Gamma(\frac{i\lambda}{2}+\frac{n+1}{2})\Gamma(-\frac{i\lambda}{2}+\frac{n+1}{2})}{\Gamma(i\lambda)\Gamma(-i\lambda)}\\ &= [(\tfrac{\lambda}{2})^2+(\tfrac{n}{2}-\tfrac{1}{2})^2]\ldots[(\tfrac{\lambda}{2})^2+(\tfrac{1}{2})^2]\,\frac{\Gamma(\frac{i\lambda}{2}+\frac{1}{2})\Gamma(-\frac{i\lambda}{2}+\frac{1}{2})}{\Gamma(i\lambda)\Gamma(-i\lambda)}\\ &= [(\tfrac{\lambda}{2})^2+(\tfrac{n}{2}-\tfrac{1}{2})^2]\ldots[(\tfrac{\lambda}{2})^2+(\tfrac{1}{2})^2]\,\frac{\pi}{\mathrm{ch}\,\frac{\pi\lambda}{2}}\,\frac{\lambda\,\mathrm{sh}\,\pi\lambda}{\pi}\\ &= 2\lambda\,\mathrm{sh}\,\tfrac{\pi\lambda}{2}\prod_{k=1}^{n/2}[(\tfrac{\lambda}{2})^2+(k-\tfrac{1}{2})^2].\end{aligned}$$

Finally, one obtains the following expression:

$$[c(2\lambda)c(-2\lambda)]^{-1} = 2^{-2n}\,[(\tfrac{n}{2}-1)!]^{-2}\,\pi^2\,\frac{\operatorname{th}\pi\lambda}{\pi\lambda}\,\lambda^2\prod_{k=2}^{n/2}[\lambda^2+(k-\tfrac{1}{2})^2].$$

We deduce then that the poles of $\lambda \mapsto [c(2\lambda)c(-2\lambda)]^{-1}$ are simple and belong to the set $\{\pm\frac{i}{2}\} \cup \{\pm i(\rho+1+k), k \in \mathbb{N}\}$. The first two of them give rise to the discrete measure $d(i) = -i\,\mathrm{Res}_{2\lambda=i}[c(2\lambda)c(-2\lambda)]^{-1}$ in (6.34). We now compute this term explicitly. Using the Eulerian expansion

$$\frac{\operatorname{th}\pi\lambda}{\pi\lambda} = \frac{2}{\pi^2}\left\{\frac{1}{\lambda^2+\frac{1}{4}} + \sum_{k=2}^{\infty}\frac{1}{\lambda^2+(k-\frac{1}{2})^2}\right\},$$

one finds

$$\mathrm{Res}_{\lambda=\frac{i}{2}}\frac{\operatorname{th}\pi\lambda}{\pi\lambda} = \frac{2}{\pi^2}\,\mathrm{Res}_{\lambda=\frac{i}{2}}\frac{1}{\lambda^2+\frac{1}{4}} = -\frac{2i}{\pi^2}.$$

On the other hand, $\prod_{k=2}^{n/2}[(\frac{i}{2})^2+(k-\frac{1}{2})^2] = (\frac{n}{2})!(\frac{n}{2}-1)!$. Hence, $d(i) = 2^{-2n-2}\,n$.

Since $[c(2\lambda)c(-2\lambda)]^{-1} = |c(2\lambda)|^{-2}$ when λ is real, we can now define the *continuous part* of the Plancherel measure by

$$d\nu(\lambda) = \frac{4}{\pi}\,\frac{d(2\lambda)}{|c(2\lambda)|^2}, \qquad (\lambda \in \mathbb{R}).$$

Theorem 6.15 (Inversion formula). *Let $p = \frac{n}{2}$. The spherical Fourier transform $\mathcal{H}$ of $\tau_p^{\pm}$-radial functions $F^{\pm} \in C_c^{\infty}(G,K,\tau_p^{\pm},\tau_p^{\pm})$ is inverted by the following formula:*

$$F^{\pm}(x) = \int_0^{\infty} d\nu(\lambda)\,\mathcal{H}^{\pm}(F^{\pm})(\lambda)\Phi^{\pm}(\lambda,x) + 2^{1-2n}\,n\,\mathcal{H}^{\pm}(F^{\pm})(\tfrac{i}{2})\Phi^{\pm}(\tfrac{i}{2},x).$$

The proof follows from the previous considerations; one should keep in mind the identity $\Phi^{\pm}(\lambda,a_t) = (\operatorname{ch}\frac{t}{2})^2\phi_{2\lambda}^{(\frac{n}{2}-1,\frac{n}{2}+1)}(\frac{t}{2})\,\mathrm{Id}$. ✓

Remark: we shall stress the link between the discrete term in the inversion formula and the contribution of the discrete series in Section 6.4.

Theorem 6.16 (Plancherel). *Let* $p = \frac{n}{2}$.

(i) *Plancherel formula: for* $F^\pm \in C_c^\infty(G, K, \tau_p^\pm, \tau_p^\pm)$,

$$\|F^\pm\|_{L^2}^2 = \frac{1}{2} C_n^{n/2} \Big\{ \int_0^\infty d\nu(\lambda)\, |\mathcal{H}^\pm(F^\pm)(\lambda)|^2 + 2^{1-2n}\, n\, |\mathcal{H}^\pm(F^\pm)(\tfrac{i}{2})|^2 \Big\}. \tag{6.35}$$

(ii) $\mathcal{H}^\pm$ *extends to a bijective isometry from the space*

$$L^2(G, K, \tau_p^\pm, \tau_p^\pm) \simeq L^2(\mathbb{R}; \tfrac{1}{2} C_n^{n/2} (2 \operatorname{sh} t)^{n-1} dt)_{even}$$

onto the space $L^2(\mathbb{R}; \frac{1}{2} C_n^{n/2} d\nu)_{even} \overset{\perp}{\oplus} \mathbb{C} \cdot \Phi^\pm(\frac{i}{2}, \cdot)$.

Moreover, this result is adaptable in an obvious manner to functions $F = F^+ + F^- \in C_c^\infty(G, K, \tau_p, \tau_p)$, *if one defines* $\mathcal{H}(F) = \frac{1}{2}\{\mathcal{H}^+(F^+) + \mathcal{H}^-(F^-)\}$.

Remark: one can check the exactness of (6.35) by testing this equality on the spherical function $\Phi^\pm(\frac{i}{2}, \cdot)$. Its L^2 norm has been calculated in (5.62). On the other hand, since

$$\mathcal{H}^\pm(\Phi^\pm(\tfrac{i}{2}, \cdot))(\tfrac{i}{2}) = \frac{2^{2n-1}}{n} \quad \text{and} \quad \mathcal{H}^\pm(\Phi^\pm(\tfrac{i}{2}, \cdot))(\lambda) = 0 \quad \forall \lambda \in \mathbb{R},$$

we see that (6.35) holds.

As in the two other cases, an alternative analysis in the Schwartz setting gives the following result (remind Corollary 5.13 (iii)).

Theorem 6.17. *Let* $p = \frac{n}{2}$. *The spherical Fourier transform* $\mathcal{H}^\pm$ *is a topological linear isomorphism between*

$$\mathcal{S}(G, K, \tau_p^\pm, \tau_p^\pm) \simeq (\operatorname{ch} t)^{-\rho}\, \mathcal{S}(\mathbb{R})_{even}$$

and the space $\mathcal{S}(\mathbb{R})_{even} \oplus \mathbb{C} \cdot \Phi^\pm(\frac{i}{2}, \cdot)$.

6.2 Construction of radial functions starting from differential forms

Our ultimate goal is to give the essential results (inversion formula, Plancherel theorem) for the Fourier transform of differential forms on $H^n(\mathbb{R})$. This will be made by applying the spherical theory described in the preceding section to certain well-chosen τ-radial functions ($\tau \in \{\tau_1, \dots, \tau_{\frac{n-1}{2}}, \tau_{\frac{n}{2}}^\pm\}$).
First, we show that there are two natural ways to construct τ-radial functions starting from functions of type τ.

Lemma 6.18. *Let f be a function of type τ. Define:*

$$F(x) = \int_G dy\, f(y) \otimes \overline{f(yx^{-1})}, \tag{6.36}$$

$$F^{\#}(x,y) = \int_K dk\, \{f(xky) \otimes \bar{\xi})\} \circ \tau(k), \quad (\xi \in \mathcal{H}_\tau \text{ fixed}). \tag{6.37}$$

Then:

(i) *F is a τ-radial function verifying*

$$\operatorname{tr} F(e) = \|f\|^2;$$

(ii) *for all $x \in G$, $F^{\#}(x,\cdot)$ is a τ-radial function verifying*

$$\operatorname{tr} F^{\#}(x,e) = \big(f(x),\xi\big)_{\mathcal{H}_\tau}\ .$$

Proof : F is τ-radial, since

$$\begin{aligned} F(k_1 x k_2) &= \int_G dy\, f(y) \otimes \overline{f(yk_2^{-1}x^{-1}k_1^{-1})} \\ &= \int_G dy\, f(yk_2) \otimes \overline{f(yx^{-1}k_1^{-1})} \\ &= \tau(k_2)^{-1} F(x) \tau(k_1)^{-1}. \end{aligned}$$

Moreover,

$$\begin{aligned} \operatorname{tr} F(e) &= \int_G dy\ \operatorname{tr}\{f(y) \otimes \overline{f(y)}\} \\ &= \int_G dy\ \big(f(y), f(y)\big)_{\mathcal{H}_\tau} \\ &= \|f\|^2. \end{aligned}$$

We have used here the identification between the vector $\xi \otimes \bar{\eta}$ of $\mathcal{H}_\tau \otimes \overline{\mathcal{H}_\tau}$ and the operator $v \mapsto (v,\eta)\xi$ of $\operatorname{End}\mathcal{H}_\tau$.

The proof of (ii) is trivial. ✓

6.3 An abstract approach of the Fourier transform

In the next paragraph, we will define concretely the Fourier transform of differential forms. We give here a theoretical justification of this definition. Again, let $\tau \in \{\tau_1, \ldots, \tau_{\frac{n-1}{2}}, \tau_{\frac{n}{2}}^{\pm}\}$.

Recall that the equivalence (3.1)

$$L^2(G) \simeq \int_{\widehat{G}}^{\oplus} d\nu(\pi)\, \mathcal{H}_\pi \mathbin{\widehat{\otimes}} \mathcal{H}_\pi^*$$

is given by the Fourier transform

$$f \longmapsto \widehat{f}(\pi) = \int_G dx\, \pi(x) f(x). \tag{6.38}$$

In accordance with the abstract Plancherel formula (Theorem 3.2), let us now define the Fourier transform $\widehat{f}(\pi) \in \mathcal{H}_\pi \otimes \mathrm{Hom}_K(\mathcal{H}_\pi, \mathcal{H}_\tau)$ when f belongs to (a dense subspace of) $L^2(G, K, \tau) = \{L^2(G) \otimes \mathcal{H}_\tau\}^K$.
First, let $f \in C_c^\infty(G) \otimes \mathcal{H}_\tau$: $f = \sum_i f_i \otimes \xi_i$, where, for each i, $f_i \in C_c^\infty(G)$. In particular, $\widehat{f_i}(\pi)$ is defined by (6.38), and we are led to put:

$$\begin{aligned}
\widehat{f}(\pi) &:= \sum_i \widehat{f_i}(\pi) \otimes \xi_i \\
&= \sum_i \Big\{ \int_G dx\, \pi(x) f_i(x) \Big\} \otimes \xi_i \\
&= \int_G dx\, \pi(x) \otimes \sum_i f_i(x) \xi_i \\
&= \int_G dx\, \pi(x) \otimes f(x).
\end{aligned}$$

Here, $\widehat{f}(\pi) \in \mathcal{L}^2(\mathcal{H}_\pi) \otimes \mathcal{H}_\tau = \mathcal{H}_\pi \mathbin{\widehat{\otimes}} \mathcal{H}_\pi^* \otimes \mathcal{H}_\tau$ [3]. Let us next check that $\widehat{f}(\pi) \in \mathcal{H}_\pi \mathbin{\widehat{\otimes}} (\mathcal{H}_\pi^* \otimes \mathcal{H}_\tau)^K \simeq \mathcal{H}_\pi \mathbin{\widehat{\otimes}} \mathrm{Hom}_K(\mathcal{H}_\pi, \mathcal{H}_\tau))$ if $f \in C_c^\infty(G, K, \tau)$, or equivalently that

$$v \mapsto \int_G dx\, \big(\pi(x) v, w\big)_{\mathcal{H}_\pi} f(x)$$

[3] $\mathcal{L}^2(\mathcal{H}_\pi)$ denotes the space of Hilbert-Schmidt operators on $\mathcal{H}_\pi$ — see [Kna86], Theorem 10.2.

defines a K-homomorphism $\widehat{f}(\pi, w)$ from $\mathcal{H}_\pi$ into $\mathcal{H}_\tau$, for every $w \in \mathcal{H}_\pi$: for any $k \in K$,

$$\begin{aligned} \widehat{f}(\pi, w)\pi(k)v &= \int_G dx\, \big(\pi(xk)v, w\big)_{\mathcal{H}_\pi}\, f(x) \\ &= \int_G dx\, \big(\pi(x)v, w\big)_{\mathcal{H}_\pi}\, f(xk^{-1}) \\ &= \tau(k)\widehat{f}(\pi, w)v. \end{aligned}$$

Let us restrict now to the representations $\pi \in \widehat{G}$ with the K-type τ, which contribute actually to the decomposition of $L^2(G, K, \tau)$. Recall that τ occurs then exactly once in π, that J^τ_π denotes a selected embedding of $\mathcal{H}_\tau$ into $\mathcal{H}_\pi$ and that $P^\tau_\pi = (J^\tau_\pi)^*$ is the corresponding orthogonal projection of $\mathcal{H}_\pi$ onto $\mathcal{H}_\tau$. Thus

$$\widehat{f}(\pi) \in \mathcal{H}_\pi \mathbin{\widehat{\otimes}} \mathrm{Hom}_K(\mathcal{H}_\pi, \mathcal{H}_\tau) = \mathcal{H}_\pi \mathbin{\widehat{\otimes}} \mathbb{C}\, P^\tau_\pi \simeq \mathcal{H}_\pi.$$

In order to identify $\widehat{f}(\pi)$ as an element of $\mathcal{H}_\pi$, we compute the trace of $\widehat{f}(\pi, w) \circ J^\tau_\pi \in \mathrm{End}\, \mathcal{H}_\tau$:

$$\begin{aligned} \mathrm{tr}\{\widehat{f}(\pi, w) \circ J^\tau_\pi\} &= \sum_j \big(\widehat{f}(\pi, w) \circ J^\tau_\pi \xi_j, \xi_j\big)_{\mathcal{H}_\tau} \\ &= \int_G dx \sum_j \big(\pi(x) \circ J^\tau_\pi \xi_j, w\big)_{\mathcal{H}_\pi} \big(f(x), \xi_j\big)_{\mathcal{H}_\tau} \\ &= \int_G dx \sum_j \big(f(x), \xi_j\big)_{\mathcal{H}_\tau}\, \overline{\big(P^\tau_\pi \circ \pi(x)^{-1} w, \xi_j\big)_{\mathcal{H}_\tau}} \\ &= \int_G dx\, \big(f(x), P^\tau_\pi \circ \pi(x)^{-1} w\big)_{\mathcal{H}_\tau} \\ &= \Big(\int_G dx\, \pi(x) \circ J^\tau_\pi f(x), w\Big)_{\mathcal{H}_\pi}. \end{aligned}$$

Consequently,

$$\widehat{f}(\pi) = \frac{1}{\dim \tau} \int_G dx\, \pi(x) \circ J^\tau_\pi f(x)\, \{\otimes P^\tau_\pi\} \tag{6.39}$$

as an element of $\mathcal{H}_\pi\, \{\otimes \mathrm{Hom}_K(\mathcal{H}_\pi, \mathcal{H}_\tau)\}$.

Remark: when $\pi = \pi_{\sigma,\lambda}$ is a principal series representation, J^τ_π maps $\mathcal{H}_\tau$ into $L^2(K, M, \sigma)$. Hence π acts in the compact picture, and $\widehat{f}(\pi)$ is an element of $L^2(K, M, \sigma)$.

6.4 Fourier transform of differential forms

Generic case

Let f be a differential p-form on $H^n(\mathbb{R})$. Its *Fourier transform* $\mathcal{H}(f)$ is the pair $(\mathcal{H}^p_{p-1}(f), \mathcal{H}^p_p(f))$ of $L^2(K, M, \sigma_q)$-valued functions of $\lambda \in \mathbb{R}$ (possibly $\lambda \in \mathbb{C}$) defined according to (6.39) by:

$$\mathcal{H}^p_q(f)(\lambda, k) = \frac{1}{C^p_n} \int_G dx\, \Psi^p_q(\lambda, x, k) f(x),$$

(whenever the integral converges) where $\Psi^p_q(\lambda, x, k) : \mathcal{H}_{\tau_p} \to \mathcal{H}_{\sigma_q}$ is the homomorphism defined by:

$$\begin{aligned}\Psi^p_q(\lambda, x, k)\xi &= \{\pi_{\sigma_q,\lambda}(x)\, J^p_q \xi\}(k) \\ &= c_{q,p}\, e^{-(i\lambda+\rho)H(x^{-1}k)} P_{\sigma_q}\, \tau_p(\underline{k}(x^{-1}k))^{-1}\xi,\end{aligned}$$

with $c_{q,p}$ as in (4.3). Notice that the integration can be performed on $G/K = H^n(\mathbb{R})$, since the function $x \mapsto \Psi^p_q(\lambda, x, k) f(x)$ is K-invariant on the right. Notice also that $\mathcal{H}^p_q(f)(\lambda, \cdot)$ belongs actually to $C^\infty(K, M, \sigma_q)$ if, for instance, f is smooth with compact support.

Remarks:

1. Here is the link between the Poisson type kernels Ψ^p_q and the τ_p-spherical functions Φ^p_q:

$$\begin{aligned}\Phi^p_q(\lambda, x)\xi &= P^p_q\, \pi_{\sigma_q,\lambda}(x^{-1})\, J^p_q \xi \\ &= c_{q,p} \int_K dk\, \tau_p(k)\{\pi_{\sigma_q,\lambda}(x^{-1})\, J^p_q \xi\}(k) \\ &= c_{q,p} \int_K dk\, \tau_p(k) \circ \Psi^p_q(\lambda, x^{-1}, k).\end{aligned}$$

2. We recall that $J^p_q \xi$ means actually $J^p_q(\xi, \cdot)$. Similarly, we will write, for short, $\mathcal{H}^p_q(f)(\lambda) := \mathcal{H}^p_q(f)(\lambda, \cdot)$ when there is no ambiguity — we have introduced in Section 6.1 the same notation for the spherical transform, but a τ-spherical function is always denoted with a capital letter, while a function of type τ is always denoted with a lower case one.

Theorem 6.19 (Inversion formula). *Let p be generic. The Fourier transform $\mathcal{H} = (\mathcal{H}^p_{p-1}, \mathcal{H}^p_p)$ of differential p-forms $f \in C^\infty_c(G, K, \tau_p)$ is inverted by the following formula:*

$$f(x) = C^p_n \sum_{q=p-1,p} \int_0^\infty d\nu_q(\lambda)\, P^p_q\, \pi_{\sigma_q,\lambda}(x^{-1})\, \mathcal{H}^p_q(f)(\lambda). \tag{6.40}$$

Proof : notice that it suffices to prove (6.40) at $x = e$. Indeed, if $\mathcal{L}$ stands for left translation, one has:

$$\begin{aligned} f(x) &= \{\mathcal{L}(x^{-1})f\}(e), \\ \text{and } \mathcal{H}^p_q(\mathcal{L}(x^{-1})f)(\lambda) &= \pi_{\sigma_q,\lambda}(x^{-1})\, \mathcal{H}^p_q(f)(\lambda). \end{aligned} \tag{6.41}$$

Now, with $f \in C^\infty_c(G, K, \tau_p)$ and $\xi \in \mathcal{H}_{\tau_p}$, one associates a function $F^\#$ by (6.37), with the property that $\operatorname{tr} F^\#(e,e) = \big(f(e), \xi\big)_{\wedge^p\mathbb{C}^n}$. On the other hand, according to (6.14),

$$\operatorname{tr} F^\#(e,e) = C^p_n \sum_q \int_0^\infty d\nu_q(\lambda) \mathcal{H}^p_q(F^\#(e,\cdot))(\lambda),$$

with

$$\begin{aligned} C^p_n\, \mathcal{H}^p_q(F^\#(e,\cdot))(\lambda) &= \int_G dy\ \operatorname{tr}\{F^\#(e,y)\Phi^p_q(\lambda, y^{-1})\} \\ &= \int_G dy \int_K dk\ \operatorname{tr}\{(f(ky) \otimes \bar{\xi})\tau_p(k)\Phi^p_q(\lambda, y^{-1})\}. \end{aligned}$$

But $\tau_p(k)\Phi^p_q(\lambda, y^{-1}) = \Phi^p_q(\lambda, y^{-1}k^{-1})$. As a consequence,

$$\begin{aligned} C^p_n\, \mathcal{H}^p_q(F^\#(e,\cdot))(\lambda) &= \int_G dy\ \operatorname{tr}\{(f(y) \otimes \bar{\xi})\Phi^p_q(\lambda, y^{-1})\} \\ &= \int_G dy\ \big(\Phi^p_q(\lambda, y^{-1})f(y), \xi\big)_{\wedge^p\mathbb{C}^n} \\ &= c_{q,p} \int_K dk\ \Big(\int_G dy\, \tau_p(k)\Psi^p_q(\lambda, y, k)f(y), \xi\Big)_{\wedge^p\mathbb{C}^n} \\ &= c_{q,p}\, C^p_n \int_K dk\ \big(\tau_p(k)\mathcal{H}^p_q(f)(\lambda, k), \xi\big)_{\wedge^p\mathbb{C}^n}\ . \end{aligned}$$

Thus, we obtain first:

$$f(e) = c_{q,p}\, C^p_n \sum_q \int_K dk \int_0^\infty d\nu_q(\lambda)\, \tau_p(k)\mathcal{H}^p_q(f)(\lambda, k),$$

and next, using (6.41):

$$f(x) = c_{q,p}\, C_n^p \sum_q \int_K dk \int_0^\infty d\nu_q(\lambda)\, \tau_p(k) \circ \{\pi_{\sigma_q,\lambda}(x^{-1})\mathcal{H}_q^p(f)(\lambda)\}(k)$$
$$= C_n^p \sum_q \int_0^\infty d\nu_q(\lambda)\, P_q^p\, \pi_{\sigma_q,\lambda}(x^{-1})\, \mathcal{H}_q^p(f)(\lambda).$$

The theorem is proved. ✓

Remark: one can also obtain the inversion formula (6.40) by using a more classical path (see e.g. [Hel94], [BR89], [Cam97a]). First, one writes the relation

$$\Phi_q^p(\lambda, y^{-1}x) = P_q^p\, \pi_{\sigma_q,\lambda}(x^{-1})\, \pi_{\sigma_q,\lambda}(y)\, J_q^p$$
$$= P_q^p\, \pi_{\sigma_q,\lambda}(x^{-1})\, \Psi_q^p(\lambda, x, \cdot).$$

Then, one defines the following convolution product: if $f \in C_c(G, K, \tau_p)$ and $F \in C(G, K, \tau_p, \tau_p)$,

$$(f * F)(x) = \int_G dy\, F(y^{-1}x) f(y).$$

Then, it can be shown that the following inversion formula holds:

$$f(x) = \sum_{q=p-1,p} \int_0^\infty d\nu_q(\lambda)\{f * \Phi_q^p(\lambda, \cdot)\}(x),$$

and, since

$$\{f * \Phi_q^p(\lambda, \cdot)\}(x) = C_n^p\, P_q^p\, \pi_{\sigma_q,\lambda}(x^{-1})\, \mathcal{H}_q^p(f)(\lambda), \tag{6.42}$$

(6.40) follows.

Theorem 6.20 (Plancherel). *Let p be generic.*

(i) *Plancherel formula: for $f \in C_c^\infty(G, K, \tau_p)$,*

$$\|f\|_{L^2}^2 = (C_n^p)^2 \sum_{q=p-1,p} \int_0^\infty d\nu_q(\lambda)\, \|\mathcal{H}_q^p(f)(\lambda)\|_{L^2(K,M,\sigma_q)}^2.$$

(ii) *The Fourier transform $\mathcal{H}$ of differential p-forms extends to a bijective isometry from $L^2(G, K, \tau_p)$ onto $\oplus_{q=p-1,p} L^2(\mathbb{R}_+, (C_n^p)^2 d\nu_q; L^2(K, M, \sigma_q))$.*

Proof : with $f \in C_c^\infty(G, K, \tau_p)$, one associates by (6.36) $F \in C_c^\infty(G, K, \tau_p, \tau_p)$ with $\operatorname{tr} F(e) = \|f\|_{L^2(G,K,\tau_p)}^2$. On the other hand, according to (6.14),

$$\operatorname{tr} F(e) = C_n^p \sum_q \int_0^\infty d\nu_q(\lambda) \mathcal{H}_q^p(F)(\lambda),$$

with

$$\mathcal{H}_q^p(F)(\lambda) = \frac{1}{C_n^p}\int_G dx\ \mathrm{tr}\{F(x)\Phi_q^p(\lambda, x^{-1})\}$$
$$= \frac{1}{C_n^p}\int_G dx \int_G dy\ \mathrm{tr}\{f(y)\otimes \overline{f(yx^{-1})}\circ P_q^p\, \pi_{\sigma_q,\lambda}(x)\, J_q^p\}.$$

Here, $\xi\otimes\bar\eta$ is viewed as the operator $v\mapsto (v,\eta)\xi$; therefore, for any operator T,

$$\mathrm{tr}(\xi\otimes\bar\eta\circ T) = \mathrm{tr}(\xi\otimes\overline{T^*\eta}) = (\xi, T^*\eta) = (T\xi,\eta).$$

This implies

$$\begin{aligned}
&\mathrm{tr}\{f(y)\otimes\overline{f(yx^{-1})}\circ P_q^p\,\pi_{\sigma_q,\lambda}(x)\,J_q^p\}\\
&=\big(P_q^p\,\pi_{\sigma_q,\lambda}(x)\,J_q^p f(y), f(yx^{-1})\big)_{\wedge^p\mathbb{C}^n}\\
&=\big(P_q^p\,\pi_{\sigma_q,\lambda}(x^{-1}y)\,J_q^p f(y), f(x)\big)_{\wedge^p\mathbb{C}^n}\\
&=\big(\pi_{\sigma_q,\lambda}(y)\,J_q^p f(y), \pi_{\sigma_q,\bar\lambda}(x)\,J_q^p f(x)\big)_{L^2(K,M,\sigma_q)}\,. \qquad (6.43)
\end{aligned}$$

Finally, since $\lambda\in\mathbb{R}$,

$$\mathcal{H}_q^p(F)(\lambda) = C_n^p\,\|\mathcal{H}_q^p(f)(\lambda)\|^2_{L^2(K,M,\sigma_q)}.$$

The surjectivity statement follows from a classical argument of density (see [Hel94], Ch. III, §2). ✓

Remark: we have not obtained a Paley-Wiener theorem for the Fourier transform of differential forms. However, we conjecture the following result, which is the analogue of Helgason's Paley-Wiener theorem in the scalar case ([Hel94], Ch. III, Theorem 5.1).

For $R>0$, denote by $PW_R(\mathbb{C}; C^\infty(K,M,\sigma_q))$ the space of entire functions: $\mathbb{C}\to C^\infty(K,M,\sigma_q)$ verifying an exponential type condition:

$$\forall\, D\in U(\mathfrak{k}),\ \forall\, N\in\mathbb{N},\ \exists\, C_{D,N}>0 \text{ such that}$$
$$\forall(\lambda,k)\in\mathbb{C}\times K,\ |h(\lambda,k:D)|\leqslant C_{D,N}\,(1+|\lambda|)^{-N}e^{R|\operatorname{Im}\lambda|}.$$

Now, let PW_R^0 be the set of couples $(h_{p-1},h_p)\in\oplus_{q=p-1,p}PW_R(\mathbb{C};C^\infty(K,M,\sigma_q))$ verifying furthermore the two following relations [4]:

$$P_q^p\,\pi_{\sigma_q,\lambda}(x^{-1})\,h_q(\lambda,\cdot) = P_q^p\,\pi_{\sigma_q,-\lambda}(x^{-1})\,h_q(-\lambda,\cdot),\quad (\forall\lambda\in\mathbb{C},\forall x\in G)$$
$$P_p^p\,\pi_{\sigma_p,\pm i(\rho-p)}(x^{-1})\,h_p(\pm i(\rho-p),\cdot)$$
$$= P_{p-1}^p\,\pi_{\sigma_{p-1},\pm i(\rho-p+1)}(x^{-1})\,h_{p-1}(\pm i(\rho-p+1),\cdot)\quad (\forall x\in G).$$

[4] The first one is suggested by (6.42) and the Weyl group invariance of the τ_p-spherical functions. The second one has the same signification as in Theorem 6.5 and comes also from (6.42).

Then $C_R^\infty(G,K,\tau_p)$ should be isomorphic to PW_R^0.

Special case $p = \frac{n-1}{2}$

Since the proofs are similar to the ones of the generic case, we simply give here the results. We fix $p = \frac{n-1}{2}$.

Let f be a p-form on $H^n(\mathbb{R})$. We define its *Fourier transform* $\mathcal{H}(f)$ as the triple $(\mathcal{H}^p_{p-1}(f), \mathcal{H}^p_{p,+}(f), \mathcal{H}^p_{p,-}(f))$ of $L^2(K,M,\sigma_q^{(\pm)})$-valued functions of $\lambda \in \mathbb{C}$ as in the generic case:

$$\mathcal{H}^p_{p-1}(f)(\lambda,k) = \frac{1}{C_n^p}\int_G dx\, \Psi^p_{p-1}(\lambda,x,k)f(x),$$
$$\mathcal{H}^p_{p,\pm}(f)(\lambda,k) = \frac{1}{C_n^p}\int_G dx\, \Psi^p_{p,\pm}(\lambda,x,k)f(x),$$

where $\Psi^p_{p-1}(\lambda,x,k)$ is defined as in the generic case and $\Psi^p_{p,\pm}(\lambda,x,k) : \mathcal{H}_{\tau_p} \to \mathcal{H}_{\sigma_p^\pm}$ is the homomorphism defined by:

$$\begin{aligned}\Psi^p_{p,\pm}(\lambda,x,k)\xi &= \{\pi_{\sigma_p^\pm,\lambda}(x)\, J^p_{p,\pm}\xi\}(k)\\ &= \sqrt{\tfrac{4n}{n+1}}\, e^{-(i\lambda+\rho)H(x^{-1}k)}P_{\sigma_p^\pm}\,\tau_p(\underline{k}(x^{-1}k))^{-1}\xi.\end{aligned}$$

Note that the integrations can be performed on $G/K = H^n(\mathbb{R})$.

Remark: as in the generic case, one has an integral formula relating Ψ and Φ.

Theorem 6.21 (Inversion formula). *Let $p = \frac{n-1}{2}$. The Fourier transform $\mathcal{H} = (\mathcal{H}^p_{p-1}, \mathcal{H}^p_{p,+}, \mathcal{H}^p_{p,-})$ of differential p-forms $f \in C_c^\infty(G,K,\tau_p)$ is inverted by the following formula:*

$$\begin{aligned}f(x) = C_n^p\int_0^\infty \Big\{ & d\nu_{p-1}(\lambda)\, P^p_{p-1}\,\pi_{\sigma_{p-1},\lambda}(x^{-1})\,\mathcal{H}^p_{p-1}(f)(\lambda)\\ & + \tfrac{1}{2}d\nu_p(\lambda)\big[P^p_{p,+}\,\pi_{\sigma_p^+,\lambda}(x^{-1})\,\mathcal{H}^p_{p,+}(f)(\lambda) + P^p_{p,-}\,\pi_{\sigma_p^-,\lambda}(x^{-1})\,\mathcal{H}^p_{p,-}(f)(\lambda)\big]\Big\}.\end{aligned} \tag{6.44}$$

Theorem 6.22 (Plancherel). *Let $p = \frac{n-1}{2}$.*

(i) *Plancherel formula: for $f \in C_c^\infty(G,K,\tau_p)$,*

$$\begin{aligned}\|f\|^2_{L^2} = (C_n^p)^2\int_0^\infty \Big\{ & d\nu_{p-1}(\lambda)\,\|\mathcal{H}^p_{p-1}(f)(\lambda)\|^2_{L^2(K,M,\sigma_{p-1})}\\ & + \tfrac{1}{2}d\nu_p(\lambda)\,\big[\|\mathcal{H}^p_{p,+}(f)(\lambda)\|^2_{L^2(K,M,\sigma_p^+)} + \|\mathcal{H}^p_{p,-}(f)(\lambda)\|^2_{L^2(K,M,\sigma_p^-)}\big]\Big\}.\end{aligned}$$

(ii) *The Fourier transform $\mathcal{H}$ of differential p-forms extends to a bijective isometry from $L^2(G, K, \tau_p)$ onto the Hilbert sum*

$$L^2(\mathbb{R}_+, (C_n^p)^2 d\nu_{p-1}; L^2(K, M, \sigma_{p-1})) \oplus L^2(\mathbb{R}_+, \tfrac{1}{2}(C_n^p)^2 d\nu_p; L^2(K, M, \sigma_p^+)) \oplus L^2(\mathbb{R}_+, \tfrac{1}{2}(C_n^p)^2 d\nu_p; L^2(K, M, \sigma_p^-)).$$

Remark: one can note the particularity we already observed in the abstract approach of the L^2 decomposition of $\frac{n-1}{2}$-forms (see §3), remarking that $d\nu_p^+ = d\nu_p^- = \frac{1}{2} d\nu_p$.

Special case $p = \frac{n}{2}$

Here we fix $p = \frac{n}{2}$, and σ_q will denote arbitrarily one of the equivalent representations σ_{p-1}, σ_p.

Let $f^\pm$ be a function of type $\tau_p^\pm$. Its *Fourier transform* is the $L^2(K, M, \sigma_q)$- valued function of $\lambda \in \mathbb{C}$ defined by

$$\mathcal{H}^{(\pm)}(f^{(\pm)})(\lambda, k) = \frac{2}{C_n^{n/2}} \int_G dx\, \Psi^\pm(\lambda, x, k) f^\pm(x),$$

where $\Psi^\pm(\lambda, x, k) : \mathcal{H}_{\tau_p^\pm} \to \mathcal{H}_{\sigma_q}$ $(q = p - 1$ or $q = p)$ is the homomorphism defined by:

$$\begin{aligned}\Psi^\pm(\lambda, x, k)\xi^\pm &= \{\pi_{\sigma_q,\lambda}(x)\, J^\pm \xi^\pm\}(k) \\ &= \sqrt{2}\, e^{-(i\lambda+\rho)H(x^{-1}k)} P_{\sigma_q}\, \tau_p^\pm(\underline{k}(x^{-1}k))^{-1} \xi^\pm.\end{aligned}$$

(Again, the integration can be done on $G/K = H^n(\mathbb{R})$.)

Remark: as in the previous cases, we have an integral formula relating $\Phi^\pm$ and $\Psi^\pm$.

Now, define $\Psi(\lambda, x, k) \in \mathrm{Hom}(\mathcal{H}_{\tau_p}, \mathcal{H}_{\sigma_q})$ by the condition:

$$\Psi(\lambda, x, k)_{|\mathcal{H}_{\tau_p^\pm}} = \Psi^\pm(\lambda, x, k).$$

If f is a C^∞ compactly supported p-form, f is a sum $f = f^+ + f^-$, with $f^\pm \in C_c^\infty(G, K, \tau_p^\pm)$. We naturally define then the Fourier transform $\mathcal{H}(f)$ of f by

$$\begin{aligned}\mathcal{H}(f)(\lambda, k) &= \tfrac{1}{2}\mathcal{H}^+(f^+)(\lambda, k) + \tfrac{1}{2}\mathcal{H}^-(f^-)(\lambda, k) \\ &= \frac{1}{C_n^{n/2}} \int_G dx\, \Psi(\lambda, x, k) f(x).\end{aligned}$$

The following result can be shown exactly as in the generic case, first for $\mathcal{H}^\pm$, and then for $\mathcal{H}$.

Theorem 6.23 (Inversion formula). *Let $p = \frac{n}{2}$ and $q = p - 1$ or p. The Fourier transform $\mathcal{H}$ of differential p-forms $f \in C_c^\infty(G, K, \tau_p)$ is inverted by the following formula:*

$$f(x) = C_n^{n/2} \Big\{ 2 \int_0^\infty d\nu(\lambda)\, P^\pm\, \pi_{\sigma_q,\lambda}(x^{-1}) \mathcal{H}(f)(\lambda) \tag{6.45}$$
$$+ 2^{1-2n}\, n \big[P^+\, \pi_{\sigma_q,-\frac{i}{2}}(x^{-1}) \mathcal{H}^+(f^+)(-\tfrac{i}{2}) + P^-\, \pi_{\sigma_q,-\frac{i}{2}}(x^{-1}) \mathcal{H}^-(f^-)(-\tfrac{i}{2}) \big] \Big\}.$$

We come now to the Plancherel theorem, which, in the case $p = \frac{n}{2}$, is not an immediate consequence of the inversion formula. In fact, we will need the celebrated subrepresentation theorem in order to relate scalar products associated with discrete series on one hand and principal series on the other hand.

Proceeding as in the generic case (proof of Theorem 6.20), we easily establish first the formula:

$$\|f\|_{L^2}^2 = (C_n^{n/2})^2 \Big\{ 2 \int_0^\infty d\nu(\lambda)\, \|\mathcal{H}(f)(\lambda)\|^2_{L^2(K,M,\sigma_q)}$$
$$+ 2^{1-2n}\, n \Big[\big(\mathcal{H}^+(f^+)(\tfrac{i}{2}), \mathcal{H}^+(f^+)(-\tfrac{i}{2}) \big)_{L^2(K,M,\sigma_q)}$$
$$+ \big(\mathcal{H}^-(f^-)(\tfrac{i}{2}), \mathcal{H}^-(f^-)(-\tfrac{i}{2}) \big)_{L^2(K,M,\sigma_q)} \Big] \Big\}, \tag{6.46}$$

which comes from (6.43). Using the notations of §3.3 and following (6.39), if $f^\pm \in C_c^\infty(G, K, \tau_p^\pm)$, we put

$$\mathcal{H}^\pm(f^\pm)(\pi^\pm) = \frac{1}{\dim \tau_p^\pm} \int_G dx\, \pi^\pm(x) \circ J_{\pi^\pm}^{\tau_p^\pm} f^\pm(x).$$

Claim. $\big(\mathcal{H}^\pm(f^\pm)(\frac{i}{2}, \cdot), \mathcal{H}^\pm(f^\pm)(-\frac{i}{2}, \cdot) \big)_{L^2(K,M,\sigma_q)} = \|\mathcal{H}^\pm(f^\pm)(\pi^\pm)\|^2_{\mathcal{H}_{\pi^\pm}}$.

Showing this assertion will take some preparation.

With each discrete series representation $\pi^\pm$, we associate a $\tau_p^\pm$-spherical function $\Phi_{\pi^\pm}$ defined by (B.13):

$$\Phi_{\pi^\pm}(x) = P_{\pi^\pm}^{\tau_p^\pm} \circ \pi^\pm(x^{-1}) \circ J_{\pi^\pm}^{\tau_p^\pm},$$

where $J_{\pi^\pm}^{\tau_p^\pm}$ is a selected embedding of $\mathcal{H}_{\tau_p^\pm}$ into $\mathcal{H}_{\pi^\pm}$ (recall that $\tau_p^\pm$ occurs only once in $\pi^\pm$ by Theorem A.2), and $P_{\pi^\pm}^{\tau_p^\pm} = (J_{\pi^\pm}^{\tau_p^\pm})^*$ the corresponding projection of $\mathcal{H}_{\pi^\pm}$ onto $\mathcal{H}_{\tau_p^\pm}$.

Proposition 6.24. $\Phi_{\pi^\pm} = \Phi^\pm(\varepsilon\frac{i}{2},\cdot)$, *where* $\Phi^\pm(\varepsilon\frac{i}{2},\cdot)$ *is the* $\tau_p^\pm$*-spherical function associated with the nonunitary principal series representation* $\pi_{\sigma_q,\varepsilon\frac{i}{2}}$ *for* $q = p-1$ *or* $q = p$ *and* $\varepsilon = \pm 1$.

Proof : on the one hand, $\Phi_{\pi^\pm}$ is harmonic by Corollary A.3 (recall that $\pi^\pm(\Omega) = 0$). Hence it is a square integrable harmonic $\tau_p^\pm$-spherical function.
On the other hand, we know by Corollary 5.13 that the only $\tau_p^\pm$-spherical functions verifying these conditions are precisely $\Phi^\pm(\frac{i}{2},\cdot) = \Phi^\pm(-\frac{i}{2},\cdot)$. ✓

In the sequel, $\mathcal{H}^{(K)}$ denotes the subspace of K-finite vectors in a representation space $\mathcal{H}$.

Proposition 6.25. $\mathcal{H}^{(K)}_{\pi^+} \oplus \mathcal{H}^{(K)}_{\pi^-}$ *is a (unitary)* $(\mathfrak{g},K)$*-submodule of* $\mathcal{H}^{(K)}_{\sigma_q,-\frac{i}{2}}$ *(or, equivalently, a unitary quotient of* $\mathcal{H}^{(K)}_{\sigma_q,\frac{i}{2}}$*).*

Proof : since $\Phi_{\pi^\pm} = \Phi^\pm(\varepsilon\frac{i}{2},\cdot)$, the proof of Corollary 5.14 shows that $-\frac{n}{2}$ is a *leading exponent* [5] of $\Phi_{\pi^\pm}$. Let us write it $-\frac{n}{2} = i\lambda_0 - \rho$ with $\lambda_0 = \frac{i}{2}$ and let us check that it is also a leading exponent for $\pi^\pm$. Suppose $\pi^\pm$ has a leading exponent $i\lambda - \rho$ greater than $-\frac{n}{2}$: then $i\lambda - i\lambda_0 \in \mathbb{N}^*$ i.e. $i\lambda \in \frac{1}{2} + \mathbb{N}$. Since $\pi^\pm$ is a discrete series, all its leading exponents $i\lambda-\rho$ must verify the L^2 condition $\mathrm{Re}(i\lambda-\rho) < -\rho = -\frac{n-1}{2}$. But if $i\lambda \in \frac{1}{2}+\mathbb{N}$, $\mathrm{Re}(i\lambda-\rho) > -\frac{n}{2} > -\rho$, which ensures that $i\lambda_0 - \rho = -\frac{n}{2}$ is actually a leading exponent for $\pi^\pm$.
By the subrepresentation theorem (Theorem 8.37 of [Kna86]), this implies that $\mathcal{H}^{(K)}_{\pi^\pm}$ is a (unitary) $(\mathfrak{g},K)$-submodule of the (nonunitary) principal series module $L^2(G, MA\overline{N}, \sigma\otimes e^{i\lambda_0}\otimes 1)^{(K)}$ for some $\sigma \in \hat{M}$. But, if w denotes as in §3 the nontrivial element of $W(\mathfrak{g},\mathfrak{a})$, one has

$$\begin{aligned} L^2(G, MA\overline{N}, \sigma\otimes e^{i\lambda_0}\otimes 1) &\simeq L^2(G, w^{-1}MANw, \sigma\otimes e^{i\lambda_0}\otimes 1)\\ &\simeq L^2(G, MAN, w\sigma\otimes e^{iw\lambda_0}\otimes 1)\\ &\simeq L^2(G, MAN, \sigma\otimes e^{-i\lambda_0}\otimes 1). \end{aligned}$$

Hence $\mathcal{H}^{(K)}_{\pi^\pm}$ is infinitesimally equivalent to a submodule of $\mathcal{H}^{(K)}_{\sigma,-\frac{i}{2}}$ for some $\sigma \in \hat{M}$.
On the other hand, we know that $\pi^\pm$ contains the K-type $\tau_p^\pm$ with multiplicity one, and that σ_q is the only possible $\sigma \in \hat{M}$ for which $\tau_p^\pm$ may occur in $\mathcal{H}_{\sigma,-\frac{i}{2}|K} \simeq L^2(K,M,\sigma)$.

[5] See [Kna86], §VIII.8.

Lastly, we recall the existence of a natural duality between $\mathcal{H}_{\sigma_q,-\frac{i}{2}}$ and $\mathcal{H}_{\sigma_q,\frac{i}{2}}$ (see (8.114) of [Kna86]) which gives the second part of the assertion. ✓

Remark: a different proof of Proposition 6.25 can be found in [BS80].

We now come to the justification of our claim.

Proposition 6.26. $\big(\mathcal{H}^\pm(f^\pm)(\frac{i}{2},\cdot),\mathcal{H}^\pm(f^\pm)(-\frac{i}{2},\cdot)\big)_{L^2(K,M,\sigma_q)} = \|\mathcal{H}^\pm(f^\pm)(\pi^\pm)\|^2_{\mathcal{H}_{\pi^\pm}}$.

Proof : in order to lighten notations, set $\sigma = \sigma_q$ and $(\tau,\pi) = (\tau_p^+,\pi^+)$ or (τ_p^-,π^-). By the preceding proposition, there exists a $(\mathfrak{g},K)$-embedding $J^\pi_{\sigma,-\frac{i}{2}}$ of $\mathcal{H}^{(K)}_\pi$ into $\mathcal{H}^{(K)}_{\sigma,-\frac{i}{2}}$. As usual, we select isometric embeddings J^τ_σ and J^τ_π of $\mathcal{H}_\tau$ into $L^2(K,M,\sigma)$ and $\mathcal{H}_\pi$. Since $\mathrm{Hom}_K(\mathcal{H}_\tau, L^2(K,M,\sigma))$ is one dimensional, we have

$$J^\tau_\sigma = J^\pi_{\sigma,-\frac{i}{2}} \circ J^\tau_\pi$$

up to a nonzero constant that we can include in $J^\pi_{\sigma,-\frac{i}{2}}$. Consider the corresponding projectors $P^\tau_\sigma = (J^\tau_\sigma)^*$ of $L^2(K,M,\sigma)$ onto $\mathcal{H}_\tau$, $P^\tau_\pi = (J^\tau_\pi)^*$ of $\mathcal{H}_\pi$ onto $\mathcal{H}_\tau$, and $P^\pi_{\sigma,\frac{i}{2}} = (J^\pi_{\sigma,-\frac{i}{2}})^*$ of $\mathcal{H}^{(K)}_{\sigma,\frac{i}{2}}$ onto $\mathcal{H}^{(K)}_\pi$. Let us check that

$$\mathcal{H}(f)(-\tfrac{i}{2}) = J_{\sigma,-\frac{i}{2}}\,\mathcal{H}(f)(\pi), \tag{6.47}$$

$$\mathcal{H}(f)(\pi) = P^\pi_{\sigma,\frac{i}{2}}\,\mathcal{H}(f)(\tfrac{i}{2}), \tag{6.48}$$

for every K-finite $f \in C_c^\infty(G,K,\tau)$. The K-finiteness assumption serves to reduce expressions to the finite dimensional setting and avoid globalization problems for $J_{\sigma,-\frac{i}{2}}$ or $P^\pi_{\sigma,\frac{i}{2}}$.

On one hand,

$$\begin{aligned}(\dim\tau)\,\mathcal{H}(f)(-\tfrac{i}{2}) &= \int_G dx\,\pi_{\sigma,-\frac{i}{2}}(x)\,J^\tau_\sigma f(x)\\ &= \int_G dx\,\pi_{\sigma,-\frac{i}{2}}(x)\,J^\pi_{\sigma,-\frac{i}{2}}\,J^\tau_\pi f(x)\\ &= \int_G dx\,J^\pi_{\sigma,-\frac{i}{2}}\,\pi(x)\,J^\tau_\pi f(x)\\ &= (\dim\tau)\,J^\pi_{\sigma,-\frac{i}{2}}\,\mathcal{H}(f)(\pi).\end{aligned}$$

On the other hand,

$$\begin{aligned}(\dim\tau)\, P^{\pi}_{\sigma,\frac{i}{2}}\, \mathcal{H}(f)(\tfrac{i}{2}) &= \int_G dx\, P^{\pi}_{\sigma,\frac{i}{2}}\, \pi_{\sigma,\frac{i}{2}}(x)\, J^{\tau}_{\sigma} f(x)\\ &= \int_G dx\, \pi(x)\, P^{\pi}_{\sigma,\frac{i}{2}}\, J^{\tau}_{\sigma} f(x)\\ &= \int_G dx\, \pi(x)\, J^{\tau}_{\pi} f(x)\\ &= (\dim\tau)\, \mathcal{H}(f)(\pi)\end{aligned}$$

since

$$P^{\pi}_{\sigma,\frac{i}{2}} \circ J^{\tau}_{\sigma} = J^{\tau}_{\pi}.$$

This identity is established as follows. The left-hand side is a K-homomorphism of $\mathcal{H}_\tau$ into $\mathcal{H}_\pi$, hence a multiple of J^{τ}_{π}:

$$P^{\pi}_{\sigma,\frac{i}{2}} \circ J^{\tau}_{\sigma} = c\, J^{\tau}_{\pi}.$$

One obtains $c = 1$ by applying P^{τ}_{π}:

$$\begin{aligned}c\,\mathrm{Id} &= c\, P^{\tau}_{\pi} \circ J^{\tau}_{\pi}\\ &= P^{\tau}_{\pi} \circ P^{\pi}_{\sigma,\frac{i}{2}} \circ J^{\tau}_{\sigma}\\ &= (J^{\pi}_{\sigma,-\frac{i}{2}} \circ J^{\tau}_{\pi})^* \circ J^{\tau}_{\sigma}\\ &= P^{\tau}_{\sigma} \circ J^{\tau}_{\sigma}\\ &= \mathrm{Id}\,.\end{aligned}$$

Finally, (6.47) and (6.48) lead to

$$\begin{aligned}\big(\mathcal{H}(f)(\tfrac{i}{2}), \mathcal{H}(f)(-\tfrac{i}{2})\big)_{L^2(K,M,\sigma)} &= \big(\mathcal{H}(f)(\tfrac{i}{2}), J_{\sigma,-\frac{i}{2}}\, \mathcal{H}(f)(\pi)\big)_{L^2(K,M,\sigma)}\\ &= \big(P^{\pi}_{\sigma,\frac{i}{2}}\, \mathcal{H}(f)(\tfrac{i}{2}), \mathcal{H}(f)(\pi)\big)_{\mathcal{H}_\pi}\\ &= \|\mathcal{H}(f)(\pi)\|^2_{\mathcal{H}_\pi},\end{aligned}$$

and this resulting identity extends to all $f \in C^\infty_c(G, K, \tau)$ by density. ✓

Note : the idea of the proof of the preceding result was kindly communicated to me by Roberto Camporesi. His original proof was based on the fact that the intertwining operator $P^{\pi}_{\sigma,\frac{i}{2}}$ is in fact a Szegö map, which can be precisely determined by a variant of the subrepresentation theorem used in Proposition 6.25 (for details, see [KW76] and [KW80]).

Finally, (6.46) and Proposition 6.26 lead to the following statement, which is in accordance with the abstract decomposition given in Theorem 3.2.

Theorem 6.27 (Plancherel). *Let* $p = \frac{n}{2}$.

(i) *Plancherel formula: for* $f \in C_c^\infty(G, K, \tau_p)$ *and* $q = p - 1$ *or* $q = p$,

$$\|f\|_{L^2}^2 = (C_n^{n/2})^2 \Big\{ 2 \int_0^\infty d\nu(\lambda)\, \|\mathcal{H}(f)(\lambda)\|_{L^2(K,M,\sigma_q)}^2 + 2^{1-2n}\, n \Big[\|\mathcal{H}^+(f^+)(\pi^+)\|_{\mathcal{H}_\pi^+}^2 + \|\mathcal{H}^-(f^-)(\pi^-)\|_{\mathcal{H}_\pi^-}^2 \Big] \Big\}.$$

(ii) *The Fourier transform* $\mathcal{H}$ *of differential* p*-forms extends to a bijective isometry from* $L^2(G, K, \tau_p)$ *onto* $2\, L^2(\mathbb{R}_+, (C_n^{n/2})^2 d\nu; L^2(K, M, \sigma_q)) \overset{\perp}{\oplus} \mathcal{H}_{\pi^+} \overset{\perp}{\oplus} \mathcal{H}_{\pi^-}$.

7 Abel transform of radial functions associated with $\wedge^p H^n(\mathbb{R})$

Our aim in this section is to study the Abel transform of τ-radial functions, when $\tau = \tau_p$ $(1 \leqslant p \leqslant \frac{n-1}{2})$ or $\tau = \tau^{\pm}_{\frac{n}{2}}$. For a given function $F \in C_c(G, K, \tau, \tau)$, we define its *Abel transform* $\mathcal{A}(F)$ by [1]

$$\mathcal{A}(F)(t) = e^{\rho t} \int_N dn\, F(a_t n) = e^{-\rho t} \int_N dn\, F(n a_t) \quad \in \mathrm{End}_M \mathcal{H}_\tau \quad (t \in \mathbb{R}). \qquad (7.1)$$

As in the case of scalar radial functions (i.e. $p = 0$, see [Koo84]), we will show that the spherical Fourier transform $\mathcal{H}$ is simply the composition $\mathcal{F} \circ \mathcal{A}$ of the Abel transform $\mathcal{A}$ followed by the Euclidean Fourier transform $\mathcal{F}$. Moreover, using a similar factorization for the Jacobi transform, we will get explicit expressions of the Abel transform in terms of differential and integral operators, and, as a consequence, an inversion formula [2]. These results will be helpful in Section 8, where we shall express the heat kernel for differential forms on $H^n(\mathbb{R})$.

We recall that the Abel transform is a special instance of the so-called *Radon transform*, which can be defined in our setting by

$$\mathcal{R}(f)(x) = e^{\rho H(x)} \int_N dn\, f(xn)$$

for any $f \in C_c(G, K, \tau)$. Let us mention that the vector-valued Radon transform has been studied in [BOS94] when G has one conjugacy class of Cartan subgroups.

[1] This transform was introduced and studied by Harish-Chandra ([HC57, HC58a, HC58b, HC66], etc.). See also the historical reference [God52], §16. It can be defined for general τ-radial (or K-central) functions on a semisimple Lie group, is often denoted by $f \mapsto F_f$, and is sometimes called the *Harish-Chandra transform*, although this vocable is more traditionaly used for the spherical Fourier transform.

[2] In the case of rank one semisimple Lie groups, other methods lead also to the inversion of the Abel transform of radial functions: see [Tak63], [LR82] and [Rou83].

7.1 Explicit expressions for the Abel transform

We consider separately the usual three cases.

Generic case

Let $F \in C_c^\infty(G, K, \tau_p, \tau_p)$. Since $\mathcal{A}(F)(t) \in \operatorname{End}_M \mathcal{H}_{\tau_p}$, we put, for $q = p-1, p$,

$$\mathcal{A}(F)(t) := \mathcal{A}_q^p(F)(t)\,\mathrm{Id} \quad \text{on } \mathcal{H}_{\sigma_q}, \tag{7.2}$$

where $\mathcal{A}_q^p(F) \in C_c^\infty(\mathbb{R})_{even}$ can be viewed as a 'partial Abel transform'. In other words, $\mathcal{A}_q^p(F)(t) = (1/C_{n-1}^q)\operatorname{tr}\{\mathcal{A}(F)(t) \circ P_{\sigma_q}\}$.

We begin by showing the relation existing between the spherical Fourier transform and the Abel transform of F. With the usual notations,

$$\begin{aligned}
\mathcal{H}_q^p(F)(\lambda) &= \frac{1}{C_n^p}\int_G dx\ \operatorname{tr}\{F(x)\Phi_q^p(\lambda, x^{-1})\} \\
&= \frac{c_{q,p}^2}{C_n^p}\int_G dx \int_K dk\, e^{-(i\lambda+\rho)H(x^{-1}k)} \operatorname{tr}\{F(x) \circ \tau_p(k) \circ P_{\sigma_q} \circ \tau(\underline{k}(x^{-1}k)^{-1})\} \\
&= \frac{1}{C_{n-1}^q}\int_G dx\, e^{-(i\lambda+\rho)H(x^{-1})} \operatorname{tr}\{F(x) \circ P_{\sigma_q} \circ \tau(\underline{k}(x^{-1})^{-1})\}.
\end{aligned}$$

But, if $x = \tilde{n}(x)\, a_t\, \tilde{k}(x) \in NAK$, then $x^{-1} = \underline{k}(x^{-1})\, a_{-t}\, \underline{n}(x^{-1}) \in KAN$, with $\underline{k}(x^{-1})^{-1} = \tilde{k}(x)$. Thus, since $dx = e^{-2\rho t}\, dn\, dt\, dk$ in the decomposition $G = NAK$,

$$\begin{aligned}
\mathcal{H}_q^p(F)(\lambda) &= \frac{1}{C_{n-1}^q}\int_N dn \int_{\mathbb{R}} dt \int_K dk\, e^{(i\lambda-\rho)t} \operatorname{tr}\{F(na_tk) \circ P_{\sigma_q} \circ \tau(k)\} \\
&= \frac{1}{C_{n-1}^q}\int_{\mathbb{R}} dt\, e^{i\lambda t} \operatorname{tr}\{\mathcal{A}(F)(t) \circ P_{\sigma_q}\} \\
&= \int_{\mathbb{R}} dt\, e^{i\lambda t}\, \mathcal{A}_q^p(F)(t) \\
&= (\mathcal{F} \circ \mathcal{A}_q^p)(F)(\lambda),
\end{aligned} \tag{7.3}$$

where $\mathcal{F}$ is the Euclidean Fourier transform defined by $\mathcal{F}(f)(\lambda) = \int_{\mathbb{R}} dt\, e^{i\lambda t} f(t)$.

The next result gives a precise expression for the transforms $\mathcal{A}_q^p$ in terms of the so-called *Weyl fractional integral operators* (see [Koo84], Section 5.3). We first introduce some notations and recall some basic facts.

For $\operatorname{Re} m > 0$ and $\varepsilon > 0$, define the Weyl operator $\mathcal{W}_m^\varepsilon$ by

$$\mathcal{W}_m^\varepsilon(f)(t) = \frac{1}{\Gamma(m)}\int_t^\infty d(\operatorname{ch}\varepsilon s)\,(\operatorname{ch}\varepsilon s - \operatorname{ch}\varepsilon t)^{m-1} f(s). \tag{7.4}$$

Then $\mathcal{W}_m^\varepsilon \circ \mathcal{W}_{m'}^\varepsilon = \mathcal{W}_{m+m'}^\varepsilon$ and $\mathcal{W}_m^\varepsilon$ maps $C_c^\infty(\mathbb{R})_{even}$ into itself. Moreover, since $\mathcal{W}_m^\varepsilon = (-\frac{d}{d(\operatorname{ch}\varepsilon t)})^j \circ \mathcal{W}_{m+j}^\varepsilon = \mathcal{W}_{m+j}^\varepsilon \circ (-\frac{d}{d(\operatorname{ch}\varepsilon t)})^j$ for $j \in \mathbb{N}$ and $\operatorname{Re} m > -j$, the Weyl transform has an analytic continuation to all complex m, $\mathcal{W}_m^\varepsilon$ has inverse $\mathcal{W}_{-m}^\varepsilon$ and it is a bijection of $C_c^\infty(\mathbb{R})_{even}$ onto itself.

Now, for complex α, β with $\alpha \neq -1, -2, \ldots$, we define an operator on $C_c^\infty(\mathbb{R})_{even}$ by

$$\mathcal{A}^{(\alpha,\beta)} = 2^{\frac{m_1-m_2}{2}}\, \pi^{\frac{m_1+m_2}{2}}\, \mathcal{W}^1_{\frac{m_1}{2}} \circ \mathcal{W}^2_{\frac{m_2}{2}}\,, \tag{7.5}$$

where $m_1 = 2(\alpha-\beta)$ and $m_2 = 2\beta+1$. We shall call this operator the *scalar Abel transform.* Then, the Jacobi transform defined in (6.2) is the composition

$$\mathcal{J}^{(\alpha,\beta)} = \mathcal{F} \circ \mathcal{A}^{(\alpha,\beta)}. \tag{7.6}$$

(cf. [Koo84], Section 5.)

Remark: when $p = 0$, $\mathcal{J}^{(\alpha,\beta)}$ and $\mathcal{A}^{(\alpha,\beta)}$ are respectively the spherical Fourier transform and the Abel transform of radial functions on the hyperbolic spaces $H^n(\mathbb{F})$, with $\alpha = \frac{dn}{2} - 1$ and $\beta = \frac{d}{2} - 1$ ($d = \dim_{\mathbb{R}} \mathbb{F}$).

Proposition 7.1. *Let p be generic. If $F \in C_c^\infty(G, K, \tau_p, \tau_p)$ has scalar components f_{p-1} and f_p, then*

(i) *the partial Abel transforms of F can be expressed in terms of scalar Abel transforms:*

$$\mathcal{A}_p^p(F)(t) = -\frac{1}{4n}\mathcal{A}^{(\frac{n}{2},-\frac{1}{2})}\Big(\frac{1}{\operatorname{sh} s}\{f_p'(s) + p(\coth s) f_p(s) - \frac{p}{\operatorname{sh} s} f_{p-1}(s)\}\Big)(t), \tag{7.7}$$

$$\mathcal{A}_{p-1}^p(F)(t) = -\frac{1}{4n}\mathcal{A}^{(\frac{n}{2},-\frac{1}{2})}\Big(\frac{1}{\operatorname{sh} s}\{f_{p-1}'(s) + (n-p)(\coth s) f_{p-1}(s) - \frac{n-p}{\operatorname{sh} s} f_p(s)\}\Big)(t), \tag{7.8}$$

where $\mathcal{A}^{(\frac{n}{2},-\frac{1}{2})} = (2\pi)^{\frac{n+1}{2}}\, \mathcal{W}^1_{\frac{n+1}{2}}$;

(ii) $\mathcal{A}_p^p(F) \equiv 0$ *if and only if* $dF = 0$*, and in that case*

$$\mathcal{A}_{p-1}^p(F)(t) = \frac{1}{4np}\Big\{-\frac{d^2}{dt^2} + [\rho - (p-1)]^2\Big\}\mathcal{A}^{(\frac{n}{2},-\frac{1}{2})}(f_p)(t); \tag{7.9}$$

$\mathcal{A}_{p-1}^p(F) \equiv 0$ *if and only if* $d^*F = 0$*, and in that case*

$$\mathcal{A}_p^p(F)(t) = \frac{1}{4n(n-p)}\Big\{-\frac{d^2}{dt^2} + (\rho - p)^2\Big\}\mathcal{A}^{(\frac{n}{2},-\frac{1}{2})}(f_{p-1})(t). \tag{7.10}$$

Proof : straightforward; one uses the relations (7.3), (7.6) and the expressions of the spherical Fourier transforms given in Proposition 6.1. ✓

Special case $p = \frac{n-1}{2}$

Similarly to the previous case, if one defines $\mathcal{A}^p_{q(,\pm)}(F)(t) := \frac{1}{C^q_{n-1}} \operatorname{tr}\{\mathcal{A}(F)(t) \circ P_{\sigma_q^{(\pm)}}\}$, one gets the relation

$$\mathcal{H}^p_{q(,\pm)}(F) = (\mathcal{F} \circ \mathcal{A}^p_{q(,\pm)})(F). \tag{7.11}$$

On the other hand, using the expression of the spherical Fourier transform in terms of the Jacobi transform (Proposition 6.9), one finally obtains:

Proposition 7.2. *Let* $p = \frac{n-1}{2}$. *If* $F \in C_c^\infty(G, K, \tau_p, \tau_p)$ *has scalar components* f_{p-1}, f_p^+ *and* f_p^-, *and if* $\tilde{f}_p = \frac{1}{2}(f_p^+ - f_p^-)$, *then*

(i) *the partial Abel transforms of F can be expressed in terms of scalar Abel transforms:*

$$\mathcal{A}^p_{p-1}(F)(t) = -\frac{1}{4n}\mathcal{A}^{(\frac{n}{2},-\frac{1}{2})}\Big(\frac{1}{\operatorname{sh} s}\{f'_{p-1}(s) + \tfrac{n+1}{2}(\coth s) f_{p-1}(s) - \tfrac{n+1}{2}(\operatorname{sh} s)^{-1} f_p(s)\}\Big)(t), \tag{7.12}$$

$$\mathcal{A}^p_{p,\pm}(F)(t) = -\frac{1}{4n}\mathcal{A}^{(\frac{n}{2},-\frac{1}{2})}\Big(\frac{1}{\operatorname{sh} s}\{f'_p(s) + \tfrac{n-1}{2}(\coth s) f_p(s) - \tfrac{n-1}{2}(\operatorname{sh} s)^{-1} f_{p-1}(s)\}\Big)(t) \mp \frac{1}{4n}\frac{d}{dt}\mathcal{A}^{(\frac{n}{2},-\frac{1}{2})}\Big(\frac{\tilde{f}_p}{\operatorname{sh}}\Big)(t), \tag{7.13}$$

where $\mathcal{A}^{(\frac{n}{2},-\frac{1}{2})} = (2\pi)^{\frac{n+1}{2}}\,\mathcal{W}^1_{\frac{n+1}{2}}$.

(ii) $\mathcal{A}^p_{p,\pm}(F) \equiv 0$ *if and only if* $dF = 0$, *and in that case*

$$\mathcal{A}^p_{p-1}(F)(t) = \frac{1}{2n(n-1)}\Big\{-\frac{d^2}{dt^2} + 1\Big\}\mathcal{A}^{(\frac{n}{2},-\frac{1}{2})}(f_p)(t); \tag{7.14}$$

$\mathcal{A}^p_{p-1}(F) \equiv 0$ *if and only if* $d^*F = 0$, *and in that case*

$$\mathcal{A}^p_{p,\pm}(F)(t) = -\frac{1}{2n(n+1)}\frac{d^2}{dt^2}\mathcal{A}^{(\frac{n}{2},-\frac{1}{2})}(f_{p-1})(t) \mp \frac{1}{4n}\frac{d}{dt}\mathcal{A}^{(\frac{n}{2},-\frac{1}{2})}\Big(\frac{\tilde{f}_p}{\operatorname{sh}}\Big)(t). \tag{7.15}$$

Special case $p = \frac{n}{2}$

Let $p = \frac{n}{2}$, $q = p$ or $q = p - 1$, and let $F^{\pm} \in C_c^{\infty}(G, K, \tau_p^{\pm}, \tau_p^{\pm})$. Again, if we put

$$\mathrm{tr}\{\mathcal{A}(F^{\pm})(t) \circ P_{\sigma_q}\} = \mathrm{tr}\,\mathcal{A}(F^{\pm})(t) := \frac{C_{n-1}^q}{2}\mathcal{A}^{\pm}(F^{\pm})(t),$$

we obtain

$$\mathcal{H}^{\pm}(F^{\pm}) = (\mathcal{F} \circ \mathcal{A}^{\pm})(F^{\pm}). \tag{7.16}$$

Now, we know (see Section 6) that $\mathcal{H}^{\pm}(F^{\pm})(\lambda) = 2^{-3}\Big\{\mathcal{J}^{(\frac{n}{2}-1,\frac{n}{2}+1)}\big(\frac{f^{\pm}(2\cdot)}{\mathrm{ch}^2}\big)\Big\}(2\lambda)$. An easy calculation leads then to the following result.

Proposition 7.3. *Let $p = \frac{n}{2}$. If $F^{\pm} \in C_c^{\infty}(G, K, \tau_p^{\pm}, \tau_p^{\pm})$ has scalar component $f^{\pm}$, then*

$$\mathcal{A}^{\pm}(F^{\pm})(t) = \frac{1}{16}\Big\{\mathcal{A}^{(\frac{n}{2}-1,\frac{n}{2}+1)}\Big(\frac{f^{\pm}(2\cdot)}{\mathrm{ch}^2}\Big)\Big\}(\tfrac{t}{2}), \tag{7.17}$$

where $\mathcal{A}^{(\frac{n}{2}-1,\frac{n}{2}+1)} = 2^{-\frac{n+7}{2}}\pi^{\frac{n-1}{2}}\mathcal{W}_{-2}^{1} \circ \mathcal{W}_{\frac{n+3}{2}}^{2}$.

Remark: the three previous propositions are still valid when F is a τ-radial function in the Schwartz space (introduced first in §6.1)

$$\mathcal{S}(G, K, \tau, \tau) = \big\{F \in C^{\infty}(G, K, \tau, \tau)\ :\ \forall D_1, D_2 \in U(\mathfrak{g}),\ \forall N \in \mathbb{N},\\ \sup_{t \geqslant 0} \|F(D_1 : a_t : D_2)\|_{\mathrm{End}\,\mathcal{H}_\tau}\,(1+t)^N e^{\rho\, t} < +\infty\big\},$$

where

$$f(X_1 \dots X_m : x : Y_1 \dots Y_n) =\\ \frac{\partial}{\partial s_1}\Big|_0 \cdots \frac{\partial}{\partial s_m}\Big|_0 \frac{\partial}{\partial t_1}\Big|_0 \cdots \frac{\partial}{\partial t_n}\Big|_0 f(\exp s_1 X_1 \cdots \exp s_m X_m \cdot x \cdot \exp t_1 Y_1 \cdots \exp t_n Y_n),$$

for any $X_1, \dots, X_m, Y_1, \dots, Y_n \in \mathfrak{g}$. Indeed, recall from [Koo84],§6 that:

$$\mathcal{A}^{(\alpha,\beta)} : (\mathrm{ch}\, t)^{-\sigma}\,\mathcal{S}(\mathbb{R})_{even} \xrightarrow{\simeq} (\mathrm{ch}\, t)^{-\sigma+\alpha+\beta+1}\,\mathcal{S}(\mathbb{R})_{even}$$

for $\alpha \geqslant \beta \geqslant -\frac{1}{2}$ and $\sigma \geqslant \alpha + \beta + 1$.

7.2 Inversion of the Abel transform

Since the partial Abel transforms are a combination of differential operators and Weyl operators — which are themselves either integral or differential —, it is quite easy to get inversion formulæ. We shall give them also in the Schwartz setting.

Generic case

According to Proposition 7.1, it suffices to determine the inverses of the scalar Abel transform $\mathcal{A}^{(\frac{n}{2},-\frac{1}{2})}$ and of the differential operator $-\frac{d^2}{dt^2}+(\rho-q)^2$.

Lemma 7.4.
(I) *Let $f\in C_c^\infty(\mathbb{R})_{even}$ or $f\in\mathcal{S}(\mathbb{R})_{even}$. Then the inverse of $\mathcal{A}^{(\frac{n}{2},-\frac{1}{2})}$ is given by*

$$\mathcal{A}^{(\frac{n}{2},-\frac{1}{2})^{-1}}(f)(t)=(2\pi)^{-\frac{n+1}{2}}\mathcal{W}^1_{-\frac{n+1}{2}}(f)(t)$$

$$=\begin{cases}(2\pi)^{-\frac{n+1}{2}}\Big(-\dfrac{d}{d(\operatorname{ch}t)}\Big)^{\frac{n+1}{2}}f(t) & \text{if } n \text{ is odd;}\\ 2^{-\frac{n+1}{2}}\pi^{-\frac{n}{2}-1}\displaystyle\int_t^\infty\frac{ds}{\sqrt{\operatorname{ch}s-\operatorname{ch}t}}\Big(-\frac{d}{ds}\Big)\Big(-\frac{d}{d(\operatorname{ch}s)}\Big)^{\frac{n}{2}}f(s) & \text{if } n \text{ is even;}\end{cases}$$

(II) *Suppose $q\neq\rho$. Then*

- (i) *the operator $-\frac{d^2}{dt^2}+(\rho-q)^2$ is an automorphism of $\mathcal{S}(\mathbb{R})_{even}$;*
- (ii) *the operator $-\frac{d^2}{dt^2}+(\rho-q)^2$ is an isomorphism between $C_c^\infty(\mathbb{R})_{even}$ and the subspace of functions $h\in C_c^\infty(\mathbb{R})_{even}$ verifying the three equivalent conditions*

$$\text{(a)}\quad \mathcal{F}(h)(\pm\lambda_q)=0;$$

$$\text{(b)}\quad \int_{\mathbb{R}}dt\,e^{\pm(\rho-q)t}\,h(t)=0;$$

$$\text{(c)}\quad \int_{\mathbb{R}}dt\,\operatorname{ch}(\rho-q)t\,h(t)=0,$$

where $\lambda_q=i(\rho-q)$ as in Theorem 6.5;

- (iii) *when defined, the inverse of $-\frac{d^2}{dt^2}+(\rho-q)^2$ is the integral operator*

$$T_q(h)(t)=\frac{1}{2(\rho-q)}\int_{\mathbb{R}}ds\,e^{-(\rho-q)|s|}h(s+t).$$

Proof: for (I), one uses the fact that the Weyl operator $\mathcal{W}^\varepsilon_{-j}$ is simply the differential operator $(-\frac{d}{d(\operatorname{ch}\varepsilon t)})^j$ when j is a positive integer. In the second formula, we have written $\mathcal{W}^1_{-\frac{n+1}{2}}$ as $\mathcal{W}^1_{\frac{1}{2}}\circ\mathcal{W}^1_{-1}\circ\mathcal{W}^1_{-\frac{n}{2}}$ in order to use again this property.

The statements in (II) follow from the fact that $\frac{d}{dt}+c$ $(c\neq 0)$ is inverted by the operator R_c given by

$$R_c(h)(t)=\begin{cases}\int_0^\infty ds\, e^{-cs}h(t-s) & \text{if } c>0,\\ -\int_0^\infty ds\, e^{cs}h(t+s) & \text{if } c<0.\end{cases}$$

Note that (i) is also obvious when one considers the Fourier transform of $\frac{d}{dt}+c$. ✓

Theorem 7.5. *Let p be generic. Suppose $F\in C_c^\infty(G,K,\tau_p,\tau_p)$ or $F\in\mathcal{S}(G,K,\tau_p,\tau_p)$ and denote by $g_{p-1}=\mathcal{A}^p_{p-1}(F)$, $g_p=\mathcal{A}^p_p(F)$ the scalar components of $\mathcal{A}(F)$. Then we have the following inversion formulæ:*

$$\begin{aligned} f_{p-1}(t)&=4n(n-p)\,\{\mathcal{A}^{(\frac{n}{2},-\frac{1}{2})^{-1}}\circ T_p\}(g_p)(t)\\ &\qquad+4np\,(\tfrac{\operatorname{sh}t}{p}\tfrac{d}{dt}+\operatorname{ch}t)\,\{\mathcal{A}^{(\frac{n}{2},-\frac{1}{2})^{-1}}\circ T_{p-1}\}(g_{p-1})(t),\\ f_p(t)&=4np\,\{\mathcal{A}^{(\frac{n}{2},-\frac{1}{2})^{-1}}\circ T_{p-1}\}(g_{p-1})(t)\\ &\qquad+4n(n-p)\,(\tfrac{\operatorname{sh}t}{n-p}\tfrac{d}{dt}+\operatorname{ch}t)\,\{\mathcal{A}^{(\frac{n}{2},-\frac{1}{2})^{-1}}\circ T_p\}(g_p)(t),\end{aligned}$$

the explicit expressions of $\mathcal{A}^{(\frac{n}{2},-\frac{1}{2})^{-1}}$ and T_q having been determined in Lemma 7.4. As a consequence, the Abel transform $\mathcal{A}=(\mathcal{A}^p_{p-1},\mathcal{A}^p_p)$ is a topological linear isomorphism

(i) *from $\mathcal{S}(G,K,\tau_p,\tau_p)$ onto $\mathcal{S}(\mathbb{R})_{even}\oplus\mathcal{S}(\mathbb{R})_{even}$, and*

(ii) *(for every $R>0$) from $C_R^\infty(G,K,\tau_p,\tau_p)$ onto the subspace of couples (h_{p-1},h_p) $\in C_R^\infty(\mathbb{R})_{even}\oplus C_R^\infty(\mathbb{R})_{even}$ verifying the condition*

$$\mathcal{F}(h_{p-1})(\pm\lambda_{p-1})=\mathcal{F}(h_p)(\pm\lambda_p)\qquad(\lambda_q=i(\rho-q)),$$

i.e. $\int_{\mathbb{R}}dt\,e^{\pm(\rho-p+1)t}\,h_{p-1}(t)=\int_{\mathbb{R}}dt\,e^{\pm(\rho-p)t}\,h_p(t)$.

Moreover, if F is (co)closed, the inversion formulæ can be simplified:

- *if $dF=0$,*

$$\begin{aligned} f_p(t)&=4np\,\{\mathcal{A}^{(\frac{n}{2},-\frac{1}{2})^{-1}}\circ T_{p-1}\}(g_{p-1})(t),\\ f_{p-1}(t)&=(\tfrac{\operatorname{sh}t}{p}\tfrac{d}{dt}+\operatorname{ch}t)f_p(t);\end{aligned}$$

- *if $d^*F=0$,*

$$\begin{aligned} f_{p-1}(t)&=4n(n-p)\,\{\mathcal{A}^{(\frac{n}{2},-\frac{1}{2})^{-1}}\circ T_p\}(g_p)(t),\\ f_p(t)&=(\tfrac{\operatorname{sh}t}{n-p}\tfrac{d}{dt}+\operatorname{ch}t)f_{p-1}(t).\end{aligned}$$

Proof : suppose for instance $F \in \mathcal{S}(G, K, \tau_p, \tau_p)$. Arguments are then similar to the ones that work for the inversion of $\mathcal{H}$ (Theorem 6.7). If $g_q = \mathcal{A}_q^p(F)$, Proposition 7.1 shows that we have to solve the system

$$\begin{cases} f_p' + p(\coth s) f_p - \frac{p}{\operatorname{sh} s} f_{p-1} = -4n(\operatorname{sh} s)\mathcal{A}^{(\frac{n}{2},-\frac{1}{2})^{-1}}(g_p)(s), \\ f_{p-1}' + (n-p)(\coth s) f_{p-1} - \frac{n-p}{\operatorname{sh} s} f_p = -4n(\operatorname{sh} s)\mathcal{A}^{(\frac{n}{2},-\frac{1}{2})^{-1}}(g_{p-1})(s). \end{cases}$$

Suppose first $g_p \equiv 0$ (i.e. $dF = 0$). Then we get

$$\{L + p(n-p+1)\} f_p = -4np\, \mathcal{A}^{(\frac{n}{2},-\frac{1}{2})^{-1}}(g_{p-1})(s), \tag{7.18}$$

where $L = L_{\frac{n}{2},-\frac{1}{2}}$ was defined in (5.24). Using (5.55) of [Koo84], we see that

$$\mathcal{A}^{(\frac{n}{2},-\frac{1}{2})} \circ L = \left(\tfrac{d^2}{ds^2} - (\tfrac{n+1}{2})^2\right) \circ \mathcal{A}^{(\frac{n}{2},-\frac{1}{2})},$$

so that (7.18) yields

$$\left(\tfrac{d^2}{ds^2} - (\rho - p + 1)^2\right) \mathcal{A}^{(\frac{n}{2},-\frac{1}{2})}(f_p) = -4np\, g_{p-1}(s),$$

hence the inversion formula when $dF = 0$. The case $g_{p-1} \equiv 0$ (i.e. $d^*F = 0$) is handled in a similar way, and the general solution is the superposition of the two particular ones.
Furthermore $\mathcal{A}$ is continuous for the Schwartz topologies, using the relation $\mathcal{H} = \mathcal{F} \circ \mathcal{A}$ and Theorem 6.7 (it is also a consequence of Harish-Chandra's work in the group case: [HC66], §18). ✓

Special case $p = \frac{n-1}{2}$

When $p = \frac{n-1}{2}$, the method is globally the same as in the generic case. We recall the easy following facts:

- the operator $-\frac{d^2}{dt^2}$ is an isomorphism from $C_c^\infty(\mathbb{R})_{even}$, resp. $\mathcal{S}(\mathbb{R})_{even}$ onto the subspace of $C_c^\infty(\mathbb{R})_{even}$, resp. $\mathcal{S}(\mathbb{R})_{even}$ of functions h verifying the equivalent conditions

$$\text{(i)} \quad \mathcal{F}(h) = 0 = (\mathcal{F}(h))'(0),$$
$$\text{(ii)} \quad \int_{\mathbb{R}} dt\, h(t) = 0 = \int_{\mathbb{R}} dt\, t\, h(t);$$

- when defined, the inverse of $-\frac{d^2}{dt^2}$ is given by $-\int_0^\infty ds\, s\, h(t+s)$.

We have then the following result.

Theorem 7.6. *Let* $p = \frac{n-1}{2}$. *Suppose* $F \in C_c^\infty(G, K, \tau_p, \tau_p)$ *or* $F \in \mathcal{S}(G, K, \tau_p, \tau_p)$ *and denote by* $g_{p-1} = \mathcal{A}^p_{p-1}(F)$, $g_p^\pm = \mathcal{A}^p_{p,\pm}(F)$ *the scalar components of* $\mathcal{A}(F)$. *Put* $g_p = \frac{1}{2}(g_p^+ + g_p^-)$ *and* $\tilde{g}_p = \frac{1}{2}(g_p^+ - g_p^-)$. *Then we have the following inversion formulæ:*

$$\begin{aligned}
f_{p-1}(t) &= 2n(n+1)\, \mathcal{A}^{(\frac{n}{2},-\frac{1}{2})^{-1}} \circ \int_t^\infty ds\,(t-s)\, g_p(s)(s) \\
&\quad + 2n(n-1)\,(\tfrac{2\operatorname{sh} t}{n-1}\tfrac{d}{dt} + \operatorname{ch} t)\, \{\mathcal{A}^{(\frac{n}{2},-\frac{1}{2})^{-1}} \circ T_{p-1}\}(g_{p-1})(t), \\
f_p(t) &= 2n(n-1)\, \{\mathcal{A}^{(\frac{n}{2},-\frac{1}{2})^{-1}} \circ T_{p-1}\}(g_{p-1})(t) \\
&\quad + 2n(n+1)\,(\tfrac{2\operatorname{sh} t}{n+1}\tfrac{d}{dt} + \operatorname{ch} t)\, \mathcal{A}^{(\frac{n}{2},-\frac{1}{2})^{-1}} \circ \int_t^\infty ds\,(t-s)\, g_p(s), \\
\tilde{f}_p(t) &= 4n(\operatorname{sh} t)\, \mathcal{A}^{(\frac{n}{2},-\frac{1}{2})^{-1}} \circ \int_t^\infty ds\, \tilde{g}_p(s),
\end{aligned}$$

and $f_p^\pm = f_p \pm \tilde{f}_p$.
As a consequence, the Abel transform $\mathcal{A} = (\mathcal{A}^p_{p-1}, \mathcal{A}^p_{p,+}, \mathcal{A}^p_{p,-})$ *is a topological linear isomorphism*

(i) *from* $\mathcal{S}(G, K, \tau_p, \tau_p)$ *onto the subspace of triples* $(h_{p-1}, h_p^+, h_p^-) \in \mathcal{S}(\mathbb{R})_{even} \oplus \mathcal{S}(\mathbb{R}) \oplus \mathcal{S}(\mathbb{R})$ *verifying the conditions*

$$h_p^\pm(-t) = h_p^\mp(t), \tag{7.19}$$

$$\int_{\mathbb{R}} dt\, h_p^+(t) = \int_{\mathbb{R}} dt\, h_p^-(t) = \int_{\mathbb{R}} dt\, t\,\{h_p^+(t) + h_p^-(t)\} = 0. \tag{7.20}$$

(ii) *(for every* $R > 0$*) from* $C_R^\infty(G, K, \tau_p, \tau_p)$ *onto the subspace of triples* $(h_{p-1}, h_p^+, h_p^-) \in C_R^\infty(\mathbb{R})_{even} \oplus C_R^\infty(\mathbb{R}) \oplus C_R^\infty(\mathbb{R})$ *verifying (7.19) and (7.20) above, and the additional condition*

$$\mathcal{F}(h_{p-1})(\pm i) = \mathcal{F}(h_p^\pm)(0),$$

i.e. $\int_{\mathbb{R}} dt\, (\operatorname{ch} t)\, h_{p-1}(t) = \int_{\mathbb{R}} dt\, h_p^\pm(t)$.

Moreover, if F *is (co)closed, the inversion formulæ can be simplified:*

- *if* $dF = 0$,

$$\begin{aligned}
f_p^+(t) = f_p^-(t) &= 2n(n-1)\, \{\mathcal{A}^{(\frac{n}{2},-\frac{1}{2})^{-1}} \circ T_{p-1}\}(g_{p-1})(t), \\
f_{p-1}(t) &= (\tfrac{2\operatorname{sh} t}{n-1}\tfrac{d}{dt} + \operatorname{ch} t) f_p^\pm(t);
\end{aligned}$$

• *if* $d^*F = 0$,

$$f_{p-1}(t) = 2n(n+1)\,\mathcal{A}^{(\frac{n}{2},-\frac{1}{2})^{-1}} \circ \int_t^\infty ds\,(t-s) g_p(s),$$
$$\tilde{f}_p(t) = 4n(\operatorname{sh} t)\,\mathcal{A}^{(\frac{n}{2},-\frac{1}{2})^{-1}} \circ \int_t^\infty ds\,\tilde{g}_p(s),$$
$$f_p(t) = (\tfrac{2\operatorname{sh} t}{n+1}\tfrac{d}{dt} + \operatorname{ch} t) f_{p-1}(t).$$

Special case $p = \frac{n}{2}$

Here, the scalar Abel transform $\mathcal{A}^{(\frac{n}{2}-1,\frac{n}{2}+1)}$ is inverted by:

$$\mathcal{A}^{(\frac{n}{2}-1,\frac{n}{2}+1)^{-1}}(f)(t) = 2^{\frac{n+7}{2}} \pi^{-\frac{n-1}{2}}\, \mathcal{W}^2_{-\frac{n+3}{2}} \circ \mathcal{W}^1_2(f)(t)$$
$$= 2^{\frac{n+7}{2}} \pi^{\frac{n}{2}} \int_t^\infty \frac{ds}{\sqrt{\operatorname{ch} 2s - \operatorname{ch} 2t}} \Big(-\frac{d}{ds}\Big)\Big(-\frac{d}{d(\operatorname{ch} 2s)}\Big)^{\frac{n}{2}+1} \int_s^\infty d(\operatorname{ch} r)(\operatorname{ch} r - \operatorname{ch} s) f(r).$$

(We have used the factorization $\mathcal{W}^2_{-\frac{n+3}{2}} = \mathcal{W}^2_{-\frac{n}{2}-1} \circ \mathcal{W}^2_{-1} \circ \mathcal{W}^2_{\frac{1}{2}}$.)

Now, the following result easily follows.

Theorem 7.7 (Inversion formula). *Let* $p = \frac{n}{2}$.

(I) *Suppose first* $F^\pm \in C_c^\infty(G, K, \tau_p^\pm, \tau_p^\pm)$ *and denote by* $g^\pm$ *the scalar component of* $\mathcal{A}^\pm(F^\pm)$. *Then we have the inversion formula:*

$$F^\pm(a_t) = 16\,(\operatorname{ch} 2t)^2 \left\{\mathcal{A}^{(\frac{n}{2}-1,\frac{n}{2}+1)^{-1}}\big(g^\pm(2\cdot)\big)\right\}(2t)\,\mathrm{Id}\,. \tag{7.21}$$

As a consequence, the Abel transform $\mathcal{A}^\pm$ *is a topological linear isomorphism from* $C_R^\infty(G, K, \tau_p^\pm, \tau_p^\pm) \simeq C_R^\infty(\mathbb{R})_{even}$ *onto* $C_R^\infty(\mathbb{R})_{even}$ *for every* $R > 0$.

(II) *Suppose now* $F^\pm \in \mathcal{S}(G, K, \tau_p^\pm, \tau_p^\pm)$. *Then the* discrete term

$$2^{1-2n}\, n\,\mathcal{H}^\pm(F^\pm)(\tfrac{i}{2})\,\Phi^\pm(\tfrac{i}{2}, a_t) = 2^{1-2n}\, n\,\mathcal{H}^\pm(F^\pm)(\tfrac{i}{2})\,(\operatorname{ch}\tfrac{t}{2})^{-n}\,\mathrm{Id}_{\tau_p^\pm},$$

which generates the discrete part $\mathcal{S}(G, K, \tau_p^\pm, \tau_p^\pm)_d = \operatorname{Ker} \mathcal{A}^\pm$ *of* $\mathcal{S}(G, K, \tau_p^\pm, \tau_p^\pm)$, *has to be added to the right-hand side of (7.21). As a consequence,* $\mathcal{A}^\pm$ *is a topological linear isomorphism from* the continuous part

$$\mathcal{S}(G, K, \tau_p^\pm, \tau_p^\pm)_c = \{\mathbb{C} \cdot \Phi^\pm(\tfrac{i}{2}, \cdot)\}^\perp$$

onto $\mathcal{S}(\mathbb{R})_{even}$.

Proof of (II)*:* it has been shown by Harish-Chandra ([HC66], Selberg's conjecture) that the kernel of the Abel transform is exactly the discrete part of the Schwartz space. Thus the assertions in (II) follow from Theorems 6.15 & 6.17. ✓

8 The heat kernel on $\wedge^p H^n(\mathbb{R})$

Fourier analysis was developed at the origin in order to solve some classical 'P.D.E.'s', such as the *wave equation* or the *heat equation.* In this section, we will deal with the latter. First, we will express the heat kernel for differential forms on $H^n(\mathbb{R})$ by using the inversion formula for the spherical Fourier transform established in Section 6; and, secondly, we will get explicit expressions by using the inversion formula for the Abel transform established in Section 7. A third part will be devoted to the calculation of certain topological invariants, the so-called Novikov-Shubin invariants, which are defined in terms of the heat kernel on L^2 forms — they have already been determined in [Lot92], but this is also an immediate consequence of our work.

The definition of the heat kernel requires some preliminaries.

8.1 Preliminary: an operator-valued spherical Fourier transform

For convenience, suppose p generic (all calculations will be easily adaptable to the other two cases). For $F \in C_c(G, K, \tau_p, \tau_p)$, let

$$\widetilde{\mathcal{H}}_q^p(F)(\lambda) = \int_G dx\, \Phi_q^p(\lambda, x^{-1}) \circ F(x) \qquad \in \operatorname{End} \mathcal{H}_{\tau_p}$$

be the *operator-valued spherical Fourier transform* on $C_c(G, K, \tau_p, \tau_p)$.

Lemma 8.1. *The operator-valued and scalar spherical Fourier transforms are related by the (equivalent) formulæ:*

$$\widetilde{\mathcal{H}}_q^p(F)(\lambda) = \mathcal{H}_q^p(F)(\lambda)\ \mathrm{Id},$$
$$\mathcal{H}_q^p(F)(\lambda) = \frac{1}{C_n^p} \operatorname{tr} \widetilde{\mathcal{H}}_q^p(F)(\lambda).$$

Proof : for all $k \in K$, using the fact that $P_q^p \in \operatorname{Hom}_K(L^2(K, M, \sigma_q), \mathcal{H}_{\tau_p})$,

$$\begin{aligned}\tau_p(k) \circ \widetilde{\mathcal{H}}_q^p(F)(\lambda) \circ \tau_p(k^{-1}) &= \int_G dx\, \tau_p(k) \circ P_q^p \circ \pi_{\sigma_q,\lambda}(x) \circ J_q^p \circ F(x) \circ \tau_p(k^{-1}) \\ &= \int_G dx\, P_q^p \circ \pi_{\sigma_q,\lambda}(kx) \circ J_q^p \circ F(kx) \\ &= \widetilde{\mathcal{H}}_q^p(F)(\lambda).\end{aligned}$$

Thus, Schur's lemma implies that $\widetilde{\mathcal{H}}_q^p(F)(\lambda)$ is a multiple (depending on λ) of the identity on $\mathcal{H}_{\tau_p}$. Taking traces, we see immediately that this multiple is precisely $\mathcal{H}_q^p(F)(\lambda)$. ✓

The next result is primordial for defining the heat kernel. If $f \in C_c(G, K, \tau_p)$ and $F \in C_c(G, K, \tau_p, \tau_p)$, the convolution product of f with F is the function in $C_c(G, K, \tau_p)$ defined by

$$(f * F)(x) = \int_G dy\, F(y^{-1}x) f(y).$$

Proposition 8.2. *If $f \in C_c(G, K, \tau_p)$ and $F \in C_c(G, K, \tau_p, \tau_p)$, we have:*

$$\mathcal{H}_q^p(f * F)(\lambda, k) = \mathcal{H}_q^p(F)(\lambda) \cdot \mathcal{H}_q^p(f)(\lambda, k).$$

Proof : let us drop the variable k in the Fourier transform of functions of type τ_p. We have:

$$\begin{aligned}(\dim \tau_p)\, \mathcal{H}_q^p(f * F)(\lambda) &= \int_G dx\, \pi_{\sigma_q,\lambda}(x) \circ J_q^p \Big(\int_G dy\, F(y^{-1}x) f(y) \Big) \\ &= \int_G dy\, \pi_{\sigma_q,\lambda}(y) \circ I(\lambda)\, f(y), \qquad (8.1)\end{aligned}$$

where $I(\lambda) = \int_G dx\, \pi_{\sigma_q,\lambda}(x) \circ J_q^p \circ F(x)$. But, for all $k \in K$, the obvious relation $\pi_{\sigma_q,\lambda}(k) \circ I(\lambda) = I(\lambda) \circ \tau_p(k)$ shows that $I(\lambda)$ has to be a multiple (depending on λ) of the generator J_q^p of $\mathrm{Hom}_K(\mathcal{H}_{\tau_p}, L^2(K, M, \sigma_q))$:

$$I(\lambda) = \mathrm{cst}(\lambda)\, J_q^p.$$

On the other hand, the identities $P_q^p \circ I(\lambda) = \widetilde{\mathcal{H}}_q^p(F)(\lambda) = \mathrm{cst}(\lambda)\,\mathrm{Id}$ imply, by the previous lemma, that $\mathrm{cst}(\lambda) = \mathcal{H}_q^p(F)(\lambda)$. Therefore, (8.1) becomes:

$$\begin{aligned}(\dim \tau_p)\, \mathcal{H}_q^p(f * F)(\lambda) &= \mathcal{H}_q^p(F)(\lambda) \int_G dy\, \pi_{\sigma_q,\lambda}(y) \circ J_q^p f(y) \\ &= (\dim \tau_p)\, \mathcal{H}_q^p(F)(\lambda) \cdot \mathcal{H}_q^p(f)(\lambda, k),\end{aligned}$$

and the proposition is proved. ✓

8.2 The heat equation on $\wedge^p H^n(\mathbb{R})$

We recall first the definition of the Schwartz space for functions of type τ on G:

$$\mathcal{S}(G,K,\tau) = \big\{ f \in C^\infty(G,K,\tau) \,:\, \forall D_1, D_2 \in U(\mathfrak{g}),\ \forall N \in \mathbb{N}, \\ \sup_{x\in G} \|f(D_1 : x : D_2)\|_{\mathcal{H}_\tau} \,(1+d(o,x))^N e^{\rho\, d(o,x)} < +\infty \big\},$$

which is similar to the one given for τ-radial functions on G (see §7.1).

Generic case

Let $f \in \mathcal{S}(G,K,\tau_p)$ [1]. The heat equation for p-forms on $H^n(\mathbb{R})$ is the differential problem

$$\begin{cases} \dfrac{\partial}{\partial t} u(t,x) = -\Delta_x\, u(t,x), \\ \lim\limits_{t\to 0} u(t,x) = f(x), \end{cases} \tag{8.2}$$

for all $t > 0$ and $x \in G$, with $u(t,\cdot) \in \mathcal{S}(G,K,\tau_p)$. Here, Δ_x means unsurprisingly that the Laplacian is applied to the function $u(t,\cdot)$. Taking the Fourier transform $\mathcal{H}^p_q$ $(q = p, p-1)$ of each side of the equalities, we get

$$\begin{cases} \dfrac{\partial}{\partial t} \mathcal{H}^p_q(u)(t,\lambda,k) = -\{\lambda^2 + (\rho-q)^2\}\, \mathcal{H}^p_q(u)(t,\lambda,k), \\ \lim\limits_{t\to 0} \mathcal{H}^p_q(u)(t,\lambda,k) = \mathcal{H}^p_q(f)(\lambda,k), \end{cases} \tag{8.3}$$

whose solution is well-known:

$$\mathcal{H}^p_q(u)(t,\lambda,k) = e^{-t\{\lambda^2+(\rho-q)^2\}}\, \mathcal{H}^p_q(f)(\lambda,k).$$

Now, we define the *heat kernel* on $\mathcal{S} \wedge^p H^n(\mathbb{R})$ as the element $H_t \in \mathcal{S}(G,K,\tau_p,\tau_p)$ such that $\mathcal{H}^p_q(H_t)(\lambda) = e^{-t\{\lambda^2+(\rho-q)^2\}}$. Using the inversion formula for τ_p-radial functions, the heat kernel is exactly:

$$H_t(x) = \sum_{q=p-1,p} \int_0^\infty d\nu_q(\lambda)\, e^{-t\{\lambda^2+(\rho-q)^2\}}\, \Phi^p_q(\lambda,x) \quad (x \in G,\ t>0). \tag{8.4}$$

Hence, applying Proposition 8.2 (trivially extended to the Schwartz setting), the solution of the heat equation (8.2) can be written as:

$$u(t,x) = (f * H_t)(x).$$

---

[1] The initial condition may be taken in other function spaces, like $L^2(G,K,\tau_p)$,...

Special case $p = \frac{n-1}{2}$

If $p = \frac{n-1}{2}$, the previous argument leads to define the heat kernel as the element $H_t \in \mathcal{S}(G, K, \tau_p, \tau_p)$ such that:

$$\mathcal{H}^p_{p-1}(H_t)(\lambda) = e^{-t(\lambda^2+1)},$$
$$\mathcal{H}^p_{p,+}(H_t)(\lambda) = \mathcal{H}^p_{p,-}(H_t)(\lambda) = e^{-t\lambda^2},$$

which implies — by Proposition 6.9 — that the scalar components h_p^+ and h_p^- of H_t are equal, and so we come back to the generic case [2]. For $x \in G$ and $t > 0$, the heat kernel is exactly given by:

$$H_t(x) = \int_0^\infty d\nu_{p-1}(\lambda)\, e^{-t(\lambda^2+1)}\, \Phi^p_{p-1}(\lambda, x) + \frac{1}{2}\int_0^\infty d\nu_p(\lambda)\, e^{-t\lambda^2}\{\Phi^p_{p,+}(\lambda, x) + \Phi^p_{p,-}(\lambda, x)\}. \tag{8.5}$$

Special case $p = \frac{n}{2}$

Here, we define the *continuous part* of the heat kernel $(H_t^\pm)_c \in \mathcal{S}(G, K, \tau_p^\pm, \tau_p^\pm)$ for $x \in G$ and $t > 0$ by:

$$(H_t^\pm)_c(x) = \int_0^\infty d\nu(\lambda)\, e^{-t(\lambda^2+\frac{1}{4})}\, \Phi^\pm(\lambda, x). \tag{8.6}$$

But, as we want to solve the heat equation in the whole Schwartz context, we must add the *discrete part*

$$H_d^\pm(x) = 2^{1-2n}\, n\, \Phi^\pm(\tfrac{i}{2}, x) \tag{8.7}$$

(which does not depend on t), since (6.45) shows that

$$\begin{aligned} f^\pm \mapsto & 2^{1-2n}\, n\, f^\pm * \Phi^\pm(\tfrac{i}{2}, \cdot) \\ &= 2^{1-2n}\, n\, C_n^{n/2}\, P^\pm \circ \pi_{\sigma_q, -\frac{i}{2}}(\cdot\cdot)^{-1}\, \mathcal{H}^\pm(f^\pm)(-\tfrac{i}{2}) \end{aligned}$$

is the projector of $L^2(G, K, \tau_p^\pm)$ onto the space of the discrete series $\mathcal{H}_{\pi^\pm}$.

[2] Note that this implies also that the partial Abel transforms $\mathcal{A}^p_{p,+}(H_t)$ and $\mathcal{A}^p_{p,-}(H_t)$ are equal.

8.3 Explicit expressions for the heat kernel on $\wedge^p H^n(\mathbb{R})$

The point is now to proceed with the usual method to express the heat kernel, that is to say to use the factorization relation $\mathcal{H} = \mathcal{F} \circ \mathcal{A}$ established in Section 7 or, more precisely its inverse. We remind the classical (and basic) fact

$$\mathcal{F}^{-1}(e^{-t(\lambda^2+c)})(r) = \tfrac{1}{\sqrt{4\pi t}}\, e^{-c\,t-\frac{r^2}{4t}}$$

for $r \in \mathbb{R}$ and any complex constant c, so that we know precisely the expressions of the Abel transforms of the heat kernel. For instance, in the generic case,

$$\mathcal{A}^p_q(H_t)(r) = \tfrac{1}{\sqrt{4\pi t}} e^{-t(\rho-q)^2-\frac{r^2}{4t}}.$$

Hence, by applying Theorems 7.5, 7.6 and 7.7, we get rather explicit expressions for H_t. Note that, in our setting, we must add a discrete term to the inversion formula when $p = \frac{n}{2}$. Namely,

$$H_t(a_r) = (H_t^+)_c(a_r) + (H_t^-)_c(a_r) + H_d^+(a_r) + H_d^-(a_r),$$

with

$$\begin{aligned}
(H_t^\pm)_c(a_r) &= 16(\operatorname{ch} 2t)^2 \big(\mathcal{A}^{(\frac{n}{2}-1,\frac{n}{2}+1)^{-1}}(\tfrac{1}{\sqrt{4\pi t}} e^{-\frac{t}{4}-\frac{r^2}{t}})\big)(2t)\, \mathrm{Id}_{\tau^\pm_{\frac{n}{2}}},\\
\text{and}\quad H_d^\pm(a_r) &= 2^{1-2n}\, n\, \Phi^\pm(\tfrac{i}{2}, a_r)\\
&= 2^{1-2n}\, n\, (\operatorname{ch} \tfrac{r}{2})^{-n}\, \mathrm{Id}_{\tau^\pm_{\frac{n}{2}}}.
\end{aligned}$$

Another method can give us slightly simpler expressions for the heat kernel. It consists in remarking that dd^*H_t and d^*dH_t are respectively closed and coclosed. Therefore, we can use the (simpler) inversion formulæ for the Abel transform we obtained with these assumptions. Then, it suffices to notice that, by definition,

$$\begin{aligned}
H_t(x) = -\int_t^\infty dr\, (dd^*H_r + d^*dH_r)(x),&\\
\left[+2^{1-2n}\, n\, \{\Phi^+(\tfrac{i}{2}, x) + \Phi^-(\tfrac{i}{2}, x)\} \quad \text{if } p = \tfrac{n}{2}\right]&
\end{aligned}$$

since $\lim_{t\to\infty} H_t(x) = 0$, resp. $\lim_{t\to\infty}(H_t)_c(x) = 0$ by (8.4), (8.5), resp. by (8.6).

REMARK: getting sharp estimates for the heat semi-group on Riemannian symmetric spaces of the noncompact type (and more general Riemannian manifolds) has been an intensive subject of study: among many references, let us mention in the rank-one case [LR82], [Ank88], [DM88], [LR96], and in the general case [Ank92], [CGM93], [AJ]. As far as we know, nothing similar is known for the case of bundles over these spaces. We believe that our results of Section 7.2 can be used to produce such estimates.

8.4 The Novikov-Shubin invariants

The *Novikov-Shubin invariant* of a closed oriented differentiable manifold M is a real (positive) number $\alpha_p(M)$ related to the decay at infinity of the heat kernel $e^{-t\Delta_p}$ on the L^2 p-forms on the universal covering of M. More precisely, with our notations,

$$\alpha_p(M) = \sup\{\beta_p \,:\, \operatorname{tr} H_t(e) \underset{t\to+\infty}{=} O(t^{-\beta_p/2})\}.$$

The invariance property of $\alpha_p(M)$ with respect to the differentiable structure of M was proved in [NS86] and [ES]. Lott improved considerably this result by showing that $\alpha_p(M)$ is in fact defined for all closed oriented topological manifolds, and is a topological invariant (see [Lot92]). In the same article was shown the following result — using [Mia79, Mia80] —, for which we are able to give a quick proof.

Proposition 8.3. *Let M be a (closed oriented) real hyperbolic manifold, with universal covering $H^n(\mathbb{R})$. Then*

$$\alpha_p(M) = \begin{cases} +\infty & \text{if } p \neq \frac{n\pm1}{2}, \frac{n}{2}, \\ 1 & \text{if } p = \frac{n\pm1}{2}, \\ 0 & \text{if } p = \frac{n}{2}. \end{cases}$$

Proof : first, it is obvious that $\alpha_p(M) = 0$ when $p = \frac{n}{2}$, using (8.7). Suppose now p generic (the proof is similar in the last case). Using (8.4), we have

$$H_t(e) = \sum_q e^{-t(\rho-q)^2} \int_0^\infty d\nu_q(\lambda)\, e^{-t\lambda^2}\, \mathrm{Id}\,.$$

Call $\eta_q(t)$ the integral in the last expression. Assume $t \geqslant 0$ and put $\mu = t^{\frac{1}{2}}\,\lambda$. Then, using the expression of $d\nu_q$ given in Section 6, we get

$$\eta_q(t) = \mathrm{cst} \int_0^\infty d\mu\, \frac{t^{-\frac{1}{2}}\, e^{-\mu^2}}{\{\mu^2 t^{-1} + (\rho-q)^2\}|c(\mu t^{-\frac{1}{2}})|^2}.$$

Now, we recall from Section 6 that (for real values of μ)

- ∘ if n is odd, $|c(\mu t^{-\frac{1}{2}})|^{-2} = \text{cst}\ \mu^2 t^{-1} \prod_{k=1}^{(n-1)/2} (\mu^2 t^{-1} + k^2)$;
- ∘ if n is even, $|c(\mu t^{-\frac{1}{2}})|^{-2} = \text{cst}\ \frac{\text{th}(\pi\mu t^{-\frac{1}{2}})}{\mu t^{-\frac{1}{2}}} \mu^2 t^{-1} \prod_{k=1}^{n/2} (\mu^2 t^{-1} + (k - \frac{1}{2})^2)$.

In both cases, $|c(\mu t^{-\frac{1}{2}})|^{-2} \sim \text{cst}\, t^{-1}$ as $t \to +\infty$. Hence $\eta_q(t) \sim \text{cst}\, t^{-\frac{1}{2}}$, and it follows that

$$\operatorname{tr} H_t(e) \underset{t\to+\infty}{\sim} \text{cst}\, t^{-\frac{1}{2}} \sum_q e^{-t(\rho-q)^2} \underset{t\to+\infty}{\sim} \text{cst}\, t^{-\frac{1}{2}}\, e^{-t(\rho-p)^2}. \qquad \checkmark$$

Appendix A. Discrete series in $L^2 \wedge^p (G/K)$

In this appendix we describe the discrete series contribution to the space $L^2 \wedge^p (G/K) \equiv L^2(G, K, \tau_p)$ of square-integrable differential forms of degree p on a general Riemannian symmetric space G/K of noncompact type. We believed that this result was *folklore*, i.e. known to specialists but not explicitly written down in the literature, until Pierre Pansu pointed out recently to Jean-Philippe Anker the reference [Bor85]. Nevertheless, for the sake of simplicity and completeness, it may be worth in our opinion to devote this appendix to the elementary presentation we came up with. Our arguments are fundamentally the same as in [Bor85] and were borrowed for the most part from [BW80], Ch. II.

We need some basic informations about discrete series, which are due to Harish-Chandra and Schmid, and can be found for instance in [Kna86], §IX.7. Recall that G has discrete series if and only if G and K have equal rank. Thus let $\mathfrak{t} \subset \mathfrak{k} \subset \mathfrak{g}$ be a Cartan subalgebra. Denote by $R_K = R(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) \subset R_G = R(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ and by $W_K \subset W_G$ the corresponding root systems and Weyl groups. Fix a positive subsystem R_K^+ in R_K or, equivalently, a positive Weyl chamber $\mathfrak{h}_K^+$ with respect to K in $\mathfrak{h} = i\mathfrak{t}$. There are $r = |W_G|/|W_K|$ positive subsystems in R_G which are compatible with R_K^+, i.e. whose intersection with R_K coincides with R_K^+. Let R_G^+ be one of them and let $\mathfrak{h}_G^+$ be the corresponding positive G-chamber in $\mathfrak{h}$. We have $\overline{\mathfrak{h}_K^+} = \cup_{1 \leqslant j \leqslant r} \overline{w_j \cdot \mathfrak{h}_G^+}$ for distinguished representatives $w_1 = \mathrm{Id}, w_2, \ldots, w_r$ of $W_K \backslash W_G$ in W_G. As usual δ_K and δ_G denote the half-sums of roots in R_K^+ and R_G^+ respectively.

Proposition A.1. *Given $\Lambda \in (\mathfrak{h}^*)_G^+$ such that $\Lambda + \delta_G$ is analytically integral and $1 \leqslant j \leqslant r$, there exists a discrete series representation $\pi_{w_j \cdot \Lambda}$ of G with the following properties:*

(a) *$\pi_{w_j \cdot \Lambda}$ has infinitesimal character $\chi_\Lambda(D) = \gamma(D)(\Lambda)$, where γ denotes as usual the* Harish-Chandra homomorphism*;*

(b) $\pi_{w_j \cdot \Lambda}$ *contains with multiplicity one the* K*-type with highest weight* $\mu_j^{\min} = w_j \cdot (\Lambda + \delta_G) - 2\delta_K$*;*

(c) *if* μ *is the highest weight of another* K*-type in* $\pi_{w_j \cdot \Lambda}$*, then* $\mu - \mu_j^{\min}$ *belongs to the positive root lattice* $w_j \cdot Q_G^+$ *generated by* $w_j \cdot R_G^+$*.*

Moreover all discrete series of G *are obtained this way and they are characterized, up to equivalence, by their* Harish-Chandra parameter $w_j \cdot \Lambda$.

We can now state the main result about discrete series occurring in $L^2 \wedge^p (G/K)$.

Theorem A.2. *Let* n *be the dimension of* G/K*.*

(i) *There are no discrete series in* $L^2 \wedge^p (G/K)$ *when* $p \neq \frac{n}{2}$*.*

(ii) *The discrete series contribution to* $L^2 \wedge^{\frac{n}{2}} (G/K)$ *consists of* $\sum_{1 \leqslant j \leqslant r}^{\oplus} \pi_{w_j \cdot \delta_G}$ *i.e. of all discrete series of* G *with trivial infinitesimal character, each occurring with multiplicity one.*

A closely related result is the L^2 eigenvalue description for the Hodge-de Rham Laplacian Δ on G/K.

Corollary A.3. Δ *has no eigenvalue on* $L^2 \wedge^p (G/K)$*, except* 0 *(with infinite multiplicity) when* $p = \frac{n}{2}$*. In this case the square integrable harmonic forms on* G/K *consist exactly of the discrete series contribution to* $L^2 \wedge^{\frac{n}{2}} (G/K)$*. More precisely* $\pi_{w_j \cdot \delta_G}$ *is realized in* $L^2(G, K, \tau_{\frac{n}{2}})$ *on the null space for the Casimir operator* Ω *acting on* $L^2(G, K, \tau_{\mu_j})$*, where* τ_{μ_j} *is the multiplicity free irreducible subrepresentation of* $\tau_{\frac{n}{2}}$ *with highest weight* $\mu_j = w_j \cdot 2\delta_G - 2\delta_K$*.*

Proof of Theorem A.2: recall the abstract Plancherel decomposition

$$L^2(G, K, \tau_p) = L^2(G, K, \tau_p)_d \oplus L^2(G, K, \tau_p)_c$$

already encountered in Section 3, where the discrete part

$$L^2(G, K, \tau_p)_d \;\simeq\; \sum\nolimits_{\pi \in \widehat{G}_d}^{\oplus} d_\pi \, \mathcal{H}_\pi \,\widehat{\otimes}\, \mathrm{Hom}_K(\mathcal{H}_\pi, \wedge^p \mathfrak{p}_{\mathbb{C}}) \tag{A.1}$$

consists of discrete series of G, and the continuous part

$$\begin{aligned} L^2(G, K, \tau_p)_c \;\simeq\; & \sum\nolimits_{MAN \subset \underline{MAN} \subsetneqq G}^{\oplus} \sum\nolimits_{\sigma \in \widehat{\underline{M}}_d}^{\oplus} \int_{\underline{\mathfrak{a}}^*}^{\oplus} d\nu(\sigma, \lambda) \\ & \times L^2(G, \underline{MAN}, \sigma \otimes e^{i\lambda} \otimes 1) \,\widehat{\otimes}\, \mathrm{Hom}_K\big(L^2(K, K \cap \underline{M}, \sigma), \wedge^p \mathfrak{p}_{\mathbb{C}}\big) \end{aligned} \tag{A.2}$$

of *generalized principal series* of G. The proof therefore reduces to a K-type comparison between discrete series of G on one hand and the (co)adjoint representation τ_p of K on $\wedge^p \mathfrak{p}_{\mathbb{C}}$ ($\equiv \wedge^p \mathfrak{p}_{\mathbb{C}}^*$) on the other hand.

Let $\pi_{w_j \cdot \Lambda}$ be a discrete series representation of G. Observe first that Λ belongs to $\delta_G + P_G^+$, where $P_G^+ = \mathbb{N}\,\mu_1 + \ldots + \mathbb{N}\,\mu_\ell$ is the positive weight lattice attached to R_G^+. Since Λ is integral and lies in the positive Weyl chamber $(\mathfrak{h}^*)_G^+$, we have indeed $\Lambda = n_1\mu_1 + \ldots + n_\ell\mu_\ell$ with coefficients $n_j \in \mathbb{N}^*$. On the other hand it is well known that $\mu_1 + \ldots + \mu_\ell = \delta_G$. Thus $\Lambda' = \Lambda - \delta_G = (n_1 - 1)\mu_1 + \ldots + (n_\ell - 1)\mu_\ell$ belongs to P_G^+, as claimed. Observe moreover that, according to Proposition A.1, the K-types in $\pi_{w_j \cdot \Lambda}$ have highest weights $\mu \in w_j \cdot 2\delta_G - 2\delta_K + w_j \cdot \Lambda' + w_j \cdot Q_G^+$, $\mu_j^{\min} = w_j \cdot 2\delta_G - 2\delta_K + w_j \cdot \Lambda'$ occurring with multiplicity one.

Consider next the representation τ_p and let us restrict to $p \leqslant \frac{n}{2}$, as one can always do by duality. Since $\mathfrak{p}_{\mathbb{C}}$ decomposes into 1-dimensional root subspaces $(\mathfrak{g}_{\mathbb{C}})_\alpha$, the weights of τ_p are all possible sums of p distinct *noncompact roots* $\alpha \in R_G \setminus R_K$. In particular $\mu_j = \sum_{\alpha \in (w_j \cdot R_G^+) \setminus R_K^+} \alpha = w_j \cdot 2\delta_G - 2\delta_K$ occurs only in $\tau_{\frac{n}{2}}$, where it is clearly a multiplicity free highest weight, and all other highest weights in τ_p belong to $\mu_j - w_j \cdot Q_G^+$.

It follows from these considerations that $\pi_{w_j \cdot \Lambda}$ and τ_p have no K-type in common, except when $\Lambda = \delta_G$ and $p = \frac{n}{2}$, where the only joint factor is the multiplicity free K-type τ_{μ_j} with highest weight μ_j. This concludes the proof of Theorem A.2. ✓

Proof of Corollary A.3: recall that $-\Delta$ is realized by the action of the *Casimir operator* Ω on $C^\infty(G, K, \tau_p)$ (this fact is referred to as *Kuga's formula* in [BW80]). It is well known that Ω acts by scalars on the various components in (A.1) and (A.2), specifically:

(i) $\pi(\Omega) = 0$ for discrete series representations $\pi = \pi_{w_j \cdot \delta_G}$;

(ii) $\pi(\Omega) = (|\Lambda_\sigma|^2 - |\delta_G|^2 - |\lambda|^2)\,\mathrm{Id}$ for generalized principal series representations $\pi = \pi_{\sigma,\lambda}$, where Λ_σ is a Harish-Chandra parameter for σ.

As a consequence, $\Delta \equiv 0$ on $L^2 \wedge^p (G/K)_d \equiv L^2(G, K, \tau_p)_d$ while Δ has no eigenvalue on $L^2 \wedge^p (G/K)_c \equiv L^2(G, K, \tau_p)_c$. Furthermore, by decomposing $\tau_{\frac{n}{2}}$ into irreducible subrepresentations and analyzing the resulting factors $L^2(G, K, \tau)$ as in the proof of Theorem A.2, we see that $L^2(G, K, \tau_{\frac{n}{2}})_d = \sum_{1 \leqslant j \leqslant r}^{\oplus} L^2(G, K, \tau_{\mu_j})_d$ and that each factor $L^2(G, K, \tau_{\mu_j})_d$ reduces exactly to the discrete series $\pi_{w_j \cdot \delta_G}$. This concludes the proof of Corollary A.3. ✓

Remark: the positivity of Δ on $L^2 \wedge^p (G/K) \equiv L^2(G, K, \tau_p)$ could have been used in the proof of Theorem A.2 in order to exclude the discrete series representations $\pi_{w_j \cdot \Lambda}$ with $\Lambda \neq \delta_G$. Indeed $\pi_{w_j \cdot \Lambda}(-\Omega) = (|\delta_G|^2 - |\Lambda|^2)\,\mathrm{Id}$, with

$$|\delta_G|^2 - |\Lambda|^2 \begin{cases} = 0 & \text{if } \Lambda = \delta_G, \\ < 0 & \text{otherwise.} \end{cases}$$

Appendix B. Abstract theory of τ-spherical functions

Let G be a unimodular locally compact group, K a compact subgroup, and τ an irreducible representation of K (that is necessarily finite dimensional and can be assumed to be unitary). A function $F : G \to \operatorname{End} \mathcal{H}_\tau$ is said *τ-radial* if it verifies the property:

$$F(k_1 x k_2) = \tau(k_2^{-1}) F(x) \tau(k_1^{-1}) \qquad (\forall x \in G,\ \forall k_1, k_2 \in K).$$

We say that the triple (G, K, τ) is a *Gelfand triple* if the convolution algebra $C_c(G, K, \tau, \tau)$ of continuous τ-radial functions on G with compact support is commutative. This is a natural generalization of the classical notion of a Gelfand pair, which corresponds to the particular case where τ is the trivial representation of K. When G is a semisimple, connected, noncompact Lie group with finite centre and K is a maximal compact subgroup, Proposition 5.1 (due to Deitmar mainly) gives several characterizations of those τ for which (G, K, τ) is a Gelfand triple. In particular, notice that $(SO_e(n,1), SO(n), \tau)$ is such a triple when $\tau = \tau_p$ ($p \neq \frac{n}{2}$) or $\tau = \tau^{\pm}_{\frac{n}{2}}$.

Our main aim in this appendix consists in characterizing in several ways τ-spherical functions among τ-radial functions for a Gelfand triple, namely as:

(i) characters of the convolution algebra $C_c(G, K, \tau, \tau)$;

(ii) solutions of a functional equation;

(iii) eigenfunctions with respect to convolution with $C_c(G, K, \tau, \tau)$,

and if G is a semisimple Lie group (with the assumptions recalled above),

(iv) eigenfunctions (in a certain sense) for the algebra $\mathbb{D}(G, K, \tau)$ of left-invariant differential operators acting on smooth sections of the homogeneous vector bundle $G \times_K \mathcal{H}_\tau$ over the Riemannian symmetric space of noncompact type G/K.

We obtain thus a perfect generalization of the standard spherical function theory in the 'scalar case', i.e. when τ is trivial (see [Far82], [Ank93], [Hel94]) or one dimensional (see [Shi90, Shi94]). In particular, the τ-spherical Fourier transform appears this way as the Gelfand transform of the commutative algebra $C_c(G, K, \tau, \tau)$. At the end of the appendix, assuming again that G is semisimple, noncompact, etc., we show that all τ-spherical functions are associated with principal series representations of G, and we relate their asymptotic behaviour to generalized Harish-Chandra c-functions.

Generalized spherical functions have been studied earlier in the literature. Several authors ([God52], [War72], [Rad76], [GV88], [Min92], [Cam97a, Cam97b]) considered τ-spherical functions of trace type, which are scalar variants of ours, and may look better suited to the non-commutative setting.
Olbrich [Olb94] considered τ-spherical functions according to (iv) above and established remarkable properties of them, even in the non-commutative case. For instance, he showed that one can restrict to representations of $\mathbb{D}(G, K, \tau)$ associated with the principal series of G, and that τ-spherical functions are then Eisenstein integrals — a fact that we shall point out in Section B.2 in our particular setting. The link between generalized spherical functions and Eisenstein integrals has been studied in a more general (but still commutative) setting — see [HOW96]. Finally, Camporesi [Cam97b] has also developed a theory similar to ours, as came out recently.

B.1 Unimodular locally compact groups

In this section, we only assumes that (G, K, τ) is a Gelfand triple. We denote by $C(G; \operatorname{End}\mathcal{H}_\tau)$ the space of continuous functions on G with values in $\operatorname{End}\mathcal{H}_\tau$; $C_c(G; \operatorname{End}\mathcal{H}_\tau)$ is defined similarly. $C_c(G; \operatorname{End}\mathcal{H}_\tau)$ is an algebra for the convolution product

$$(F * H)(x) = \int_G dy\, F(y^{-1}x)H(y) = \int_G dy\, F(y)H(xy^{-1}).$$

To justify the definition of τ-spherical functions, we begin with the following result.

Lemma B.1. *Each continuous character χ of the algebra $C_c(G, K, \tau, \tau)$ can be defined by a function $\Phi \in C(G, K, \tau, \tau)$ in the following way:*

$$\chi(F) = \frac{1}{\dim \tau} \int_G dx\ \operatorname{tr}\{F(x)\Phi(x^{-1})\}. \tag{B.1}$$

Proof : the first step consists in remarking that such a character χ extends naturally to a linear form on $C_c(G;\operatorname{End}\mathcal{H}_\tau)$ by setting $\chi(F)=\chi(F^\natural)$, where $\natural$ is the 'radialization operator', i.e. the canonical projection of $C_c(G;\operatorname{End}\mathcal{H}_\tau)$ onto $C_c(G,K,\tau,\tau)$ defined by

$$F^\natural(x)=\iint_{K\times K} dk_1\,dk_2\,\tau(k_2)F(k_1xk_2)\tau(k_1).$$

Now, χ is a continuous linear form on $C_c(G;\operatorname{End}\mathcal{H}_\tau)\simeq C_c(G)\otimes\operatorname{End}\mathcal{H}_\tau$; therefore, by a classical result, there exists an $\operatorname{End}\mathcal{H}_\tau$-valued Radon measure μ on G such that $\chi(F)=\int_G \operatorname{tr}\{F(x)d\mu(x)\}$. Furthermore, μ can be chosen in the following way

$$d\mu(x)=\tfrac{1}{\dim\tau}\Phi(x^{-1})dx,$$

with $\Phi\in C(G,K,\tau,\tau)$. Let us explain why (we exclude the trivial case $\chi\equiv 0$). We start from the fact that

$$\chi(F_1{*}F_2^\natural)=\chi(F_1)\chi(F_2^\natural)=\chi(F_1)\chi(F_2), \tag{B.2}$$

for any $F_1\in C_c(G,K,\tau,\tau)$ and any $F_2\in C_c(G;\operatorname{End}\mathcal{H}_\tau)$. That is,

$$\operatorname{tr}\{\int_G d\mu(z)\,(F_1{*}F_2^\natural)(z)\}=\chi(F_1)\chi(F_2).$$

But one can always choose F_1 such that $\chi(F_1)=\dim\tau$. So, (B.2) gives:

$$\begin{aligned}\chi(F_2)&=\frac{1}{\dim\tau}\int_G \operatorname{tr}\{d\mu(z)\int_G dx\,F_1(x^{-1}z)F_2^\natural(x)\}\\ &=\frac{1}{\dim\tau}\int_G dx\ \operatorname{tr}\{(\check{\mu}{*}F_1)(x^{-1})F_2(x)^\natural\}.\end{aligned}$$

But since

$$\int_G dx\ \operatorname{tr}\{H_1(x^{-1})H_2^\natural(x)\}=\int_G dx\ \operatorname{tr}\{H_1(x^{-1})H_2(x)\}$$

for any $H_1\in C_c(G,K,\tau,\tau)$ and any $H_2\in C_c(G;\operatorname{End}\mathcal{H}_\tau)$, we finally get

$$\begin{aligned}\chi(F_2)&=\frac{1}{\dim\tau}\int_G dx\ \operatorname{tr}\{(\check{\mu}{*}F_1)^\natural(x^{-1})F_2(x)\},\\ &=\frac{1}{\dim\tau}\int_G dx\ \operatorname{tr}\{F_2(x)\Phi(x^{-1})\},\end{aligned}$$

with $\Phi=(\check{\mu}{*}F_1)^\natural\in C(G,K,\tau,\tau)$. ✓

We will say that $\Phi \in C(G,K,\tau,\tau)$ is a *τ-spherical function on G* if it is nonzero and if the map $\chi_\Phi : F \mapsto \frac{1}{\dim\tau}\int_G dx\ \mathrm{tr}\{F(x)\Phi(x^{-1})\}$ defines a (necessarily nonzero and continuous) character of $C_c(G,K,\tau,\tau)$ — note that Φ is automatically C^∞ if G is a Lie group (in the proof of the previous lemma, F_1 can be taken in C_c^∞). We will denote by $\Sigma(G,K,\tau,\tau)$ the set of τ-spherical functions on G.

We are now able to state the main result of this section.

Theorem B.2. *Let $\Phi \in C(G,K,\tau,\tau)$ with $\Phi(e) = \mathrm{Id}$. Then the following conditions are equivalent:*

(i) Φ *is τ-spherical;*

(ii) Φ *verifies one of the three equivalent functional equations:*

$$\int_K dk\,\tau(k)\Phi(xky) = \frac{\mathrm{tr}\,\Phi(y)}{\dim\tau}\Phi(x) \qquad (\forall\, x,y\in G), \tag{B.3}$$

$$\int_K dk\,\Phi(xky)\tau(k) = \frac{\mathrm{tr}\,\Phi(x)}{\dim\tau}\Phi(y) \qquad (\forall\, x,y\in G), \tag{B.4}$$

$$\int_K dk\ \mathrm{tr}\{\tau(k)\Phi(xky)\} = \frac{\mathrm{tr}\,\Phi(x)\cdot\mathrm{tr}\,\Phi(y)}{\dim\tau} \qquad (\forall\, x,y\in G); \tag{B.5}$$

(iii) Φ *is an eigenfunction for the convolution product with $C_c(G,K,\tau,\tau)$, i.e. there exists a character λ of $C_c(G,K,\tau,\tau)$ such that for all $F \in C_c(G,K,\tau,\tau)$, $F*\Phi = \Phi*F = \lambda(F)\,\Phi$.*

The above theorem is a consequence of several intermediate results. In particular, the equivalence between (i) and (ii) can be established in a slightly more general context, as will become clear in the following statement.

Proposition B.3. *Let $\Phi \in C(G;\mathrm{End}\,\mathcal{H}_\tau)$ such that $\mathrm{tr}\,\Phi(x_0) \neq 0$ for some $x_0 \in G$. Then Φ is τ-spherical if and only if Φ verifies the functional equations (B.3), (B.4) and (B.5). Moreover, in this case, $\Phi(e) = \mathrm{Id}$ and Φ verifies the additional functional equation:*

$$\iint_{K\times K} dk_1\,dk_2\,\tau(k_1)\Phi(xk_1k_2y)\tau(k_2) = \frac{1}{(\dim\tau)^2}\Phi(x)\Phi(y); \tag{B.6}$$

Proof : We begin with the easiest part, namely: if $\Phi \in \Sigma(G,K,\tau,\tau)$, then Φ verifies (B.3), (B.4) and (B.5).

Let $F_1, F_2 \in C_c(G; \operatorname{End}\mathcal{H}_\tau)$. Then $\chi_\Phi(F_1^\natural * F_2^\natural) = \chi_\Phi(F_1^\natural)\chi_\Phi(F_2^\natural)$, hence

$$\begin{aligned} 0 &= \iint_{G\times G} dx\,dy\ \operatorname{tr}\{F_1^\natural(x^{-1}y)F_2^\natural(x)\Phi(y^{-1})\} \\ &\qquad\qquad - \frac{1}{\dim\tau}\int_G dy\ \operatorname{tr}\{F_1^\natural(y)\Phi(y^{-1})\}\ \int_G dx\ \operatorname{tr}\{F_2^\natural(x)\Phi(x^{-1})\} \\ &= \iint_{G\times G} dx\,dy\Big\{\operatorname{tr}\{F_1^\natural(y)F_2^\natural(x)\Phi(y^{-1}x^{-1})\} \\ &\qquad\qquad - \frac{1}{\dim\tau}\ \operatorname{tr}\{F_1^\natural(y)\Phi(y^{-1})\}\ \operatorname{tr}\{F_2^\natural(x)\Phi(x^{-1})\}\Big\}. \end{aligned}$$

Using the definition of the operator $\natural$ and making a change of variables, we see that the integrand can be replaced by

$$\begin{aligned} \iint_{K\times K} dk_1\,dk_2\Big\{&\operatorname{tr}\{F_1(y)\tau(k_1)\tau(k_2)F_2(x)\Phi(y^{-1}k_1k_2x^{-1})\} \\ &- \frac{1}{\dim\tau}\ \operatorname{tr}\{F_1(y)\Phi(y^{-1})\}\ \operatorname{tr}\{F_2(x)\Phi(x^{-1})\}\Big\}. \end{aligned}$$

Thus, Φ is τ-spherical if and only if, for all $F_1, F_2 \in C_c(G; \operatorname{End}\mathcal{H}_\tau)$,

$$\begin{aligned} \iint_{G\times G} dx\,dy \int_K dk\ &\operatorname{tr}\{F_1(y)\tau(k)F_2(x)\Phi(y^{-1}kx^{-1})\} \\ &= \frac{1}{\dim\tau}\int_G dy\ \operatorname{tr}\{F_1(y)\Phi(y^{-1})\}\ \int_G dx\ \operatorname{tr}\{F_2(x)\Phi(x^{-1})\}. \end{aligned} \tag{B.7}$$

If Φ is τ-spherical, using (B.7) and the fact that $(F_1, F_2) \mapsto \int_G dx\ \operatorname{tr}\{F_1(x)F_2(x^{-1})\}$ is a non-degenerated bilinear form on $L^2(G; \operatorname{End}\mathcal{H}_\tau)$, we get on the one hand:

$$\int_G dy \int_K dk\,\Phi(y^{-1}kx)F_1(y)\tau(k) = \chi_\Phi(F_1)\,\Phi(x) \tag{B.8}$$

for all $F_1 \in C_c(G; \operatorname{End}\mathcal{H}_\tau)$ and all $x \in G$; similarly, (B.7) implies

$$\int_G dx \int_K dk\,\tau(k)F_2(x)\Phi(ykx^{-1}) = \chi_\Phi(F_2)\,\Phi(y) \tag{B.9}$$

for all $F_2 \in C_c(G; \operatorname{End}\mathcal{H}_\tau)$ and all $y \in G$. Now, taking traces in both sides of (B.8), we obtain

$$\int_G dy\ \operatorname{tr}\{(F_1(y)\int_K dk\,\tau(k)\Phi(y^{-1}kx)\} = \frac{1}{\dim\tau}\int_G dy\ \operatorname{tr}\{F_1(y)\Phi(y^{-1})\operatorname{tr}\Phi(x)\},$$

hence (B.3). (B.4) follows similarly from (B.9), and (B.5) is a trivial consequence of (B.3) or (B.4).

We shall now prove the converse, using three lemmas.

Lemma B.4. *Let $\Phi \in C(G; \operatorname{End} \mathcal{H}_\tau)$ such that $\operatorname{tr} \Phi(x_0) \neq 0$ for some $x_0 \in G$. If Φ verifies (B.3) and (B.4), then Φ is τ-radial.*

Proof : this is obvious: using respectively (B.3) and (B.4), we see that for any $k_1, k_2 \in K$, and for any $x \in G$,

$$\operatorname{tr} \Phi(x_0) \cdot \Phi(xk_2) = \operatorname{tr} \Phi(x_0) \cdot \tau(k_2^{-1})\Phi(x),\ \operatorname{tr} \Phi(x_0) \cdot \Phi(k_1 x) = \operatorname{tr} \Phi(x_0) \cdot \Phi(x)\tau(k_1^{-1}),$$

and the condition of nonzero trace is sufficient to conclude. ✓

Lemma B.5. *A continuous τ-radial function F on G is entirely determined by the associated scalar function $f(x) = \operatorname{tr} F(x)$. More precisely,*

$$F(x) = \dim \tau \int_K dk\, \tau(k) \operatorname{tr} F(kx) = \dim \tau \int_K dk\, \tau(k) \operatorname{tr} F(xk) \quad (\forall x \in G).$$

Proof : this is another easy fact, using Schur's orthogonality relations for τ (see [War72], vol. II, §6.1.1 for details). ✓

Lemma B.6. *The conditions (B.3), (B.4), (B.5), (B.7), (B.8) and (B.9) are all equivalent for a continuous τ-radial function Φ.*

Proof : for instance, (B.3) is equivalent to the equation

$$\int_G dy \int_K dk\ \operatorname{tr}\{\Phi(ykx)F_1(y)\tau(k)\} = \frac{1}{\dim \tau} \operatorname{tr} \Phi(x) \int_G dy\ \operatorname{tr}\{F_1(y)\Phi(y)\}$$

for all $F_1 \in C_c(G; \operatorname{End} \mathcal{H}_\tau)$. But it is straightforward to see that, for a fixed y, the function $x \mapsto \int_K dk\, \Phi(ykx)F_1(y)\tau(k)$ is τ-radial. Hence, applying the previous lemma, we see that (B.3) is equivalent to (B.8). ✓

As a consequence of these three lemmas, we get that if $\Phi \in C(G; \operatorname{End} \mathcal{H}_\tau)$ is such that $\operatorname{tr} \Phi(x_0) \neq 0$ for some $x_0 \in G$ and if Φ verifies (B.3) and (B.4), then Φ is τ-spherical. Indeed, Φ is τ-radial by Lemma B.4, hence Φ verifies (B.7), which is also equivalent to the fact that Φ is τ-spherical.

REMARK: the condition that $\operatorname{tr} \Phi(x_0) \neq 0$ for some $x_0 \in G$ is not surprising: it corresponds to the fact that $\phi \not\equiv 0$ when one identifies τ-radial functions on G with K-central functions on G of type τ by $\phi(x) = \operatorname{tr} \Phi(x)$ and by Lemma B.5 (actually, such functions ϕ are precisely the τ-spherical functions of trace type that we have mentioned in the introduction to this appendix).

Lemma B.7. *If Φ is τ-radial, continuous, nonzero and verifies (B.3) or (B.4), then Φ is normalized, i.e. $\Phi(e) = \mathrm{Id}$.*

Proof : suppose for example that Φ verifies (B.3) (the proof is similar in the other case). By τ-radiality, we have:

$$\int_K dk\, \Phi(ykxk^{-1}) = \frac{\operatorname{tr}\Phi(x)}{\dim\tau}\Phi(y). \tag{B.10}$$

Choosing $x = e$ in (B.10), we see that

$$\Phi(y) = \frac{\operatorname{tr}\Phi(e)}{\dim\tau}\Phi(y),$$

for all $y \in G$. Since Φ is nonzero, we conclude first that

$$\operatorname{tr}\Phi(e) = \dim\tau. \tag{B.11}$$

Now, choose $y = e$ in (B.10); this gives the relation

$$\int_K dk\, \Phi(kxk^{-1}) = \frac{\operatorname{tr}\Phi(x)}{\dim\tau}\Phi(e). \tag{B.12}$$

But, if we set $\Psi(x) = \int_K dk\, \Phi(kxk^{-1})$, we note that $\Psi(x)$ intertwines $\tau(k')$ with itself, for any $k' \in K$. Thus, by Schur's lemma, we conclude that $\Psi(x) = c(x)\,\mathrm{Id}$, where $c(x)$ is a constant only depending on $x \in G$. We then take the trace of both sides of (B.12), which gives the relation $c(x)\dim\tau = \operatorname{tr}\Psi(x) = \operatorname{tr}\Phi(x)$. Finally, using (B.11), necessarily $c(e) = 1$ and $\Phi(e) = \Psi(e) = \mathrm{Id}$. ✓

To complete the proof of Proposition B.3, it remains to establish (B.6). But it is an easy consequence of Lemma B.5 applied to (B.3) or (B.4). ✓

We proceed with the proof of Theorem B.2 by showing the last characterization of τ-spherical functions.

Proposition B.8. *Let $\Phi \in C(G,K,\tau,\tau)$ such that $\Phi(e) = \mathrm{Id}$. Then Φ is τ-spherical if and only if Φ is a convolution eigenfunction with respect to $C_c(G,K,\tau,\tau)$. Moreover, in this case, the eigenvalue is $\lambda(F) = \chi_\Phi(F)$, for each $F \in C_c(G,K,\tau,\tau)$.*

Proof : suppose Φ is τ-spherical. Then, for all $x \in G$,

$$\begin{aligned}(F * \Phi)(x) &= \int_G dy\, F(y)\Phi(xy^{-1}) \\ &= \int_G dy \int_K dk\, \tau(k)F(yk)\Phi(xy^{-1}) \\ &= \int_G dy \int_K dk\, \tau(k)F(y)\Phi(xky^{-1}).\end{aligned}$$

But (B.9) holds, so that

$$(F * \Phi)(x) = \chi_\Phi(F)\,\Phi(x).$$

Conversely, suppose that for each $F \in C_c(G, K, \tau, \tau)$, there exists a complex number $\lambda(F)$ such that $F * \Phi = \lambda(F)\,\Phi$. Then

$$\begin{aligned}\lambda(F)\,\dim\tau &= \operatorname{tr}(F * \Phi)(e) \\ &= \int_G dy\ \operatorname{tr}\{F(y)\Phi(y^{-1})\}.\end{aligned}$$

This shows that $F \mapsto \lambda(F) = \chi_\Phi(F)$ is a character of $C_c(G, K, \tau, \tau)$. Hence Φ is τ-spherical. ✓

We give now a natural construction of τ-spherical functions on a Gelfand triple.

Proposition B.9. *Let π be an admissible representation of G such that τ occurs with multiplicity one in $\pi_{|K}$. Define, for $x \in G$,*

$$\Phi_\pi^\tau(x) = P_\pi^\tau \circ \pi(x^{-1}) \circ J_\pi^\tau, \tag{B.13}$$

where P_π^τ is the projection of $\mathcal{H}_\pi$ onto the τ-isotypical component $\mathcal{H}_\pi(\tau) \simeq \mathcal{H}_\tau$ and $J_\pi^\tau = (P_\pi^\tau)^*$. Then Φ_π^τ is a τ-radial function on G verifying $\Phi_\pi^\tau(e) = \mathrm{Id}$ and the functional equation (B.3). Hence Φ_π^τ is τ-spherical.

Proof : put $P = P_\pi^\tau$, $J = J_\pi^\tau$ and $\Phi = \Phi_\pi^\tau$ for short. Let (ξ_i) be a basis of $\mathcal{H}_\tau$, and set

$$\Psi(x, y) := \int_K dk\,\tau(k)\Phi(xky)$$

(the left-hand side of (B.3)). We compute the matrix coefficients of $\Psi(x, y)$:

$$\begin{aligned}\Psi(x, y)_{i,j} &= \int_K dk\,[\tau(k)\Phi(xky)]_{i,j} \\ &= \sum_l \int_K dk\,\tau(k)_{i,l}\,\Phi(xky)_{l,j} \\ &= \sum_l \int_K dk\ \big(\tau(k)\xi_l, \xi_i\big)_{\mathcal{H}_\tau}\big(\pi(k^{-1})\pi(x^{-1})J\xi_j, \pi(y^{-1})^* J\xi_l\big)_{\mathcal{H}_\pi}.\end{aligned}$$

We use then (a consequence of) Schur's orthogonality relations (since $\pi(k)$ is unitary):

$$\int_K dk\ \big(\pi(k)u_1, v_1\big)_{\mathcal{H}_\pi}\big(\tau(k^{-1})v_2, u_2\big)_{\mathcal{H}_\tau} = \tfrac{1}{\dim\tau}\ \big(Pu_1, u_2\big)_{\mathcal{H}_\tau}\big(v_2, Pv_1\big)_{\mathcal{H}_\tau},$$

for $u_1, u_2 \in \mathcal{H}_\pi$ and $v_1, v_2 \in \mathcal{H}_\tau$. Hence we get

$$\begin{aligned}\Psi(x,y)_{i,j} &= \tfrac{1}{\dim\tau}\sum_l \left(\xi_l, P\circ\pi(y^{-1})^*\circ J\xi_l\right)_{\mathcal{H}_\tau}\left(P\circ\pi(x^{-1})\circ J\xi_j,\xi_i\right)_{\mathcal{H}_\tau}\\ &= \tfrac{1}{\dim\tau}\sum_l \left(\Phi(y)\xi_l,\xi_l\right)_{\mathcal{H}_\tau}\left(\Phi(x)\xi_j,\xi_i\right)_{\mathcal{H}_\tau}\\ &= \tfrac{1}{\dim\tau}[\operatorname{tr}\Phi(y)]\,\Phi(x)_{i,j}.\end{aligned}$$

Thus, applying Theorem B.2, Φ is τ-spherical. ✓

Let us indicate how the spherical Fourier transform on a Gelfand triple (that is, the Fourier transform for τ-radial functions on G) arises naturally as the Gelfand transform of the algebra $C_c(G,K,\tau,\tau)$. This is a consequence of the general theory of commutative Banach algebras (see [Rud91]).

Recall first that, by definition, the set $\Sigma(G,K,\tau,\tau)$ of τ-spherical functions can be viewed as the so-called 'Gelfand spectrum' of $C_c(G,K,\tau,\tau)$. We define the *spherical Fourier transform* $\mathcal{H}$ *on* $C_c(G,K,\tau,\tau)$ to be the Gelfand transform on $C_c(G,K,\tau,\tau)$: If $F\in C_c(G,K,\tau,\tau)$, for all $\Phi\in\Sigma(G,K,\tau,\tau)$,

$$\mathcal{H}_F(\Phi) := \chi_\Phi(F) = \frac{1}{\dim\tau}\int_G dx\ \operatorname{tr}\{F(x)\Phi(x^{-1})\}. \tag{B.14}$$

Let us endow $\Sigma(G,K,\tau,\tau)$ with the topology of uniform convergence on compact sets, so that we can define the algebra $C(\Sigma(G,K,\tau,\tau))$ of continuous functions on the set of τ-spherical functions. Then, using the fact that $\mathcal{H}_F(\Phi)=\chi_\Phi(F)$, the following result is easily obtained.

Proposition B.10. *$\mathcal{H}: F\mapsto\mathcal{H}_F$ is an algebra homomorphism from $C_c(G,K,\tau,\tau)$ into $C(\Sigma(G,K,\tau,\tau))$.*

Now, let $\Sigma^1(G,K,\tau,\tau)$ be the Gelfand spectrum of the Banach algebra $L^1(G,K,\tau,\tau)$; it is clear that this set can be identified with the subset of bounded τ-spherical functions. The weak topology induced by duality on $\Sigma^1(G,K,\tau,\tau)$ is called the Gelfand topology. Then, we have:

Proposition B.11. *On $\Sigma^1(G,K,\tau,\tau)$, the topology of uniform convergence on compact sets coincides with the Gelfand topology. As a corollary, $\mathcal{H}$ is an injective algebra homomorphism from $L^1(G,K,\tau,\tau)$ into the set $C_0(\Sigma^1(G,K,\tau,\tau))$ of continuous functions on $\Sigma^1(G,K,\tau,\tau)$ vanishing at infinity.*

REMARKS:

1. One could also generalize that part of the classical abstract spherical function theory dealing with the Bochner theorem, the Plancherel theorem and the inversion formula for $\mathcal{H}$... We have not investigated this subject here, concentrating our efforts on the 'concrete' theory (Sections 5 and 6). For some known results, see [Cam97a, Cam97b].

2. All the results we have established here (in particular, Theorem B.2) still hold when $C_c(G, K, \tau, \tau)$ is *not* commutative. But the present approach yields too poor of a theory and one must consider finite dimensional representations of $C_c(G, K, \tau, \tau)$ instead of characters, as was done in [Olb94] for the algebra $\mathbb{D}(G, K, \tau)$ when G is a semisimple Lie group.

B.2 Semisimple connected noncompact Lie groups with finite centre

Assume now that (G, K, τ) is a Gelfand triple with G a semisimple, connected, noncompact Lie group with finite centre and K a maximal compact subgroup. We have then an additional characterization of τ-spherical functions on G.

Theorem B.12. *Let* $\Phi \in C^\infty(G, K, \tau, \tau)$ *with* $\Phi(e) = \mathrm{Id}$. *Then characterizations* (i), (ii) and (iii) *of Theorem B.2 are all equivalent to*

(iv) Φ *is an eigenfunction for the algebra* $\mathbb{D}(G, K, \tau)$, *in the sense that there exists a character* $\mathcal{X}_\Phi$ *of* $\mathbb{D}(G, K, \tau)$ *such that*

$$D\Phi(\cdot)\xi = \mathcal{X}_\Phi(D)\Phi(\cdot)\xi$$

for any $D \in \mathbb{D}(G, K, \tau)$ *and for one nonzero* $\xi \in \mathcal{H}_\tau$ *(hence for all* $\xi \in \mathcal{H}_\tau$*). More precisely, the eigenvalue is* $\mathcal{X}_\Phi(D) = \frac{1}{\dim \tau} \operatorname{tr}\{D\,\Phi\}(e)$ *for each* D.

Proof : suppose first that Φ is τ-spherical: Φ is smooth and, by Proposition B.8, $F * \Phi = \chi_\Phi(F)\,\Phi$ for $F \in C_c(G, K, \tau, \tau)$. Since elements D of $\mathbb{D}(G, K, \tau)$ are left G-invariant operators, it is easily seen that $D(F*\Phi) = F*D\Phi = \chi_\Phi(F)\,D\Phi$. Evaluating at $x = e$ and taking traces, it follows that

$$\int_G dy\ \operatorname{tr} F(y)D\Phi(y^{-1}) = \frac{1}{\dim \tau}\int_G dy\ \operatorname{tr}\big(F(y)\{\operatorname{tr} D\Phi(e)\}\Phi(y^{-1})\big),$$

and, in particular,

$$D\Phi = \frac{\operatorname{tr} D\Phi(e)}{\dim \tau}\Phi. \tag{B.15}$$

The map $\mathcal{X}_\Phi : D \mapsto \frac{1}{\dim\tau}\operatorname{tr} D\Phi(e)$ is clearly a character of $\mathbb{D}(G, K, \tau)$: by applying (B.15),

$$\begin{aligned}\mathcal{X}_\Phi(D_1 \circ D_2) &= \frac{1}{\dim\tau}\operatorname{tr}\Big(\frac{1}{\dim\tau}\{\operatorname{tr} D_2\Phi(e)\}D_1\Phi(e)\Big)\\ &= \mathcal{X}_\Phi(D_1)\,\mathcal{X}_\Phi(D_2).\end{aligned}$$

To prove the converse, we first state a lemma.

Lemma B.13. *To each character $\mathcal{X}$ of the algebra $\mathbb{D}(G, K, \tau)$ corresponds at most one normalized τ-radial function Φ such that:*

$$D\Phi = \mathcal{X}(D)\Phi, \quad \forall D \in \mathbb{D}(G, K, \tau). \tag{B.16}$$

Proof : we use classical arguments (see [Hel84] for the case of functions on G/K). Suppose there exists two normalized functions $\Phi_1, \Phi_2 \in C^\infty(G, K, \tau, \tau)$ verifying (B.16) for the same character $\mathcal{X}$.
Put $\Phi = \Phi_1 - \Phi_2$; then $D\Phi(e) = 0$ for all $D \in \mathbb{D}(G, K, \tau)$. Let us show that this implies that $D\Phi(e) = 0$ for all differential operators $D \in U(\mathfrak{g})$.
One has, for all $D \in U(\mathfrak{g})$ and $k \in K$,

$$\begin{aligned}D\Phi(e) &= \Phi(e : D)\\ &= \tau(k)\Phi(k : D : k^{-1})\tau(k)^{-1}\\ &= \tau(k)\Phi(e : (\operatorname{Ad} k)D)\tau(k)^{-1}\end{aligned}$$

Therefore, $\operatorname{tr} D\Phi(e) = \operatorname{tr} D^\natural\Phi(e)$, where $D^\natural \in U(\mathfrak{g})^K$ is defined by:

$$D^\natural = \int_K dk\,(\operatorname{Ad} k)D.$$

But $U(\mathfrak{g})^K$ and $\{\operatorname{End}\mathcal{H}_\tau \otimes U(\mathfrak{g})\}^K$ coincide on $C^\infty(G, K, \tau)$, so we can see $D^\natural$ as an element of $\mathbb{D}(G, K, \tau)$. Hence $\operatorname{tr} D\Phi(e) = \operatorname{tr} D^\natural\Phi(e) = 0$ for all $D \in U(\mathfrak{g})$. This implies that $\operatorname{tr}\Phi \equiv 0$. Indeed, it can be easily shown that Φ is analytic, as a solution of the elliptic differential equation

$$\Omega_\mathfrak{p}\Phi = \{\tau(\Omega_\mathfrak{k}) + \mathcal{X}(\Omega)\}\Phi$$

involving the Casimir operator Ω. Using Lemma B.5, we finally get that $\Phi \equiv 0$. ✓

If we suppose now that there exists $\mathcal{X}$ such that $D\Phi = \mathcal{X}(D)\Phi$ for each D, using again the fact that $D(F * \Phi) = F * D\Phi$, we see that $D(F * \Phi) = \mathcal{X}(D)(F * \Phi)$.

From the uniqueness shown in the lemma, we deduce that $F * \Phi$ has to be a complex multiple $\lambda(F)\Phi$ of Φ. Moreover, evaluating in $x = e$ and taking traces, we obtain:

$$\operatorname{tr}\{(F * \Phi)(e)\} = \int_G dy \ \operatorname{tr} F(y)\Phi(y^{-1}) = \lambda(F) \dim \tau,$$

from which we conclude that $\lambda(F) = \chi_\Phi(F)$. ✓

Remark: Theorem B.12 can be stated in the more general setting of reductive groups.

We devote the end of this appendix to collect some information about τ-spherical functions on a Gelfand triple (G, K, τ) when G is a semisimple Lie group (with the usual other assumptions). More precisely, we show that the set $\Sigma(G, K, \tau, \tau)$ is entirely constituted of τ-spherical functions attached (in the meaning of (B.13)) to (minimal) principal series representations, and we relate their asymptotics with matrix-valued Harish-Chandra's c-functions. This material is essentially borrowed from [War72], [Wal75], [Ven94], [Olb94] and [Yan94], and is presented here for the sake of completeness.

Let us recall first some standard notation. If $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the decomposition of $\mathfrak{g}$ into eigenspaces for the Cartan involution θ, choose a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{p}$, set $A = \exp \mathfrak{a}$, $\log = \exp^{-1}$ and $M = Z_K(\mathfrak{a})$. Let $P = MAN$ be a *minimal* parabolic subgroup of G (recall that the number of choices of such P depends on the dimension of $\mathfrak{a}$, i.e. on the rank of G/K). Then $G = KAN$ is an Iwasawa decomposition of G, and we shall write an element

$$G \ni x = \underline{k}(x) \ \exp H(x) \, \underline{n}(x).$$

Let S be the restricted root system of the pair $(\mathfrak{g}, \mathfrak{a})$ and S_P^+ the subsystem of positive roots associated with P (i.e. with N). As usual, we put

$$\rho = \rho_P = \frac{1}{2} \sum_{\alpha \in S_P^+} m_\alpha \, \alpha,$$

and set

$$\mathfrak{a}_+^* = \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* : \langle \lambda, \alpha \rangle > 0 \ \forall \alpha \in S_P^+\},$$
$$\mathfrak{a}_-^* = \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* : \langle \lambda, \alpha \rangle < 0 \ \forall \alpha \in S_P^+\}.$$

Set also $\overline{N} = \theta(N)$. Finally, dk is the Haar measure on K normalized by $\int_K dk = 1$, and $d\overline{n}$ is the measure on $\overline{N}$ induced by the Lebesgue measure on $\overline{\mathfrak{n}} = \theta(\mathfrak{n})$.

As in Section 3, the (minimal) principal series representations of G are defined as follows: for $(\sigma, \mathcal{H}_\sigma) \in \widehat{M}$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$,

$$\pi_{\sigma,\lambda} := \operatorname{Ind}_{MAN}^G(\sigma \otimes \exp i\lambda \otimes 1)$$

acts on the Hilbert space

$$\begin{aligned}\mathcal{H}_{\sigma,\lambda} &= L^2(G, MAN, \sigma \otimes \exp i\lambda \otimes 1) \\ &:= \{f : G \to \mathcal{H}_\sigma : f(xman) = e^{-(i\lambda+\rho)(\log a)}\sigma(m)^{-1}f(x),\ f_{|K} \in L^2(K)\}\end{aligned}$$

by left translations, and on $\mathcal{H}_{\sigma,\lambda|K} \simeq L^2(K, M, \sigma)$ by

$$\pi_{\sigma,\lambda}(x)f(k) = e^{-(i\lambda+\rho)(H(x^{-1}k))}\, f(\underline{k}(x^{-1}k)) \qquad (x \in G, k \in K).$$

We use now the definition (B.13) to produce τ-spherical functions associated with a principal series representation of G.

Recall first that the principal series $\pi_{\sigma,\lambda}$ are admissible, and, for a fixed σ, irreducible for 'generic' values of λ. Since the decomposition of $\pi_{\sigma,\lambda|K}$ into K-types does not depend on λ, Deitmar's criterion (Theorem 5.1) applies and we see that $\sigma \in \widehat{M}$ can only occur in $\tau_{|M}$ with multiplicity $\leqslant 1$. In the sequel, we denote by $\widehat{M}_\tau$ the subset of $\widehat{M}$ constituted of those representations σ which occur with multiplicity one in $\tau_{|M}$. Thus, for $\sigma \in \widehat{M}_\tau$, $\operatorname{Hom}_K(\mathcal{H}_\tau, \mathcal{H}_{\sigma,\lambda}) \simeq \operatorname{Hom}_K(\mathcal{H}_\tau, L^2(K, M, \sigma))$ is one dimensional, and

$$\xi \mapsto J_\sigma^\tau \xi = \sqrt{\tfrac{\dim\tau}{\dim\sigma}}\, P_\sigma \circ \{\tau(\cdot)^{-1}\xi\}$$

(P_σ denotes the projection onto the σ-isotypical component of $\mathcal{H}_\tau$) defines an (isometric) generator of this space. Similarly, $P_\sigma^\tau = (J_\sigma^\tau)^*$ is a generator of the one dimensional space $\operatorname{Hom}_K(\mathcal{H}_{\sigma,\lambda}, \mathcal{H}_\tau)$ and is defined by

$$P_\sigma^\tau(f) = \sqrt{\tfrac{\dim\tau}{\dim\sigma}} \int_K dk\, \tau(k)\, f(k).$$

Now, for $\sigma \in \widehat{M}_\tau$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, set

$$\begin{aligned}&\Phi_\sigma^\tau(\lambda, \cdot) : G \longrightarrow \operatorname{End}\mathcal{H}_\tau \\ &\Phi_\sigma^\tau(\lambda, x) = P_\sigma^\tau \circ \pi_{\sigma,\lambda}(x^{-1}) \circ J_\sigma^\tau.\end{aligned}$$

Then $\Phi_\sigma^\tau(\lambda, \cdot)$ is τ-spherical on G by Proposition B.9 and admits the following representation as Eisenstein integral:

Proposition B.14. *For $\sigma \in \widehat{M}_\tau$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$,*

$$\Phi_\sigma^\tau(\lambda, x) = \frac{\dim \tau}{\dim \sigma} \int_K dk\, e^{-(i\lambda+\rho)(H(xk))} \tau(k) \circ P_\sigma \circ \tau(\underline{k}(xk)^{-1}). \tag{B.17}$$

In particular, Φ_σ^τ is holomorphic is the variable λ.

Proof : we have:

$$\begin{aligned}
&\big(\Phi_\sigma^\tau(\lambda, x)\xi, \eta\big)_{\mathcal{H}_\tau} \\
&= \big(\pi_{\sigma,\lambda}(x)^{-1} \circ J_\sigma^\tau(\xi, \cdot), J_\sigma^\tau(\eta, \cdot)\big)_{L^2(K,M,\sigma)} \\
&= \tfrac{\dim \tau}{\dim \sigma} \int_K dk\, e^{-(i\lambda+\rho)H(xk)} \, \big(P_\sigma\{\tau(\underline{k}(xk))^{-1}\xi\}, P_\sigma\{\tau(k)^{-1}\eta\}\big)_{\mathcal{H}_\sigma} \\
&= \tfrac{\dim \tau}{\dim \sigma} \int_K dk\, e^{-(i\lambda+\rho)H(xk)} \, \big(\tau(k) \circ P_\sigma\{\tau(\underline{k}(xk))^{-1}\xi\}, \eta\big)_{\mathcal{H}_\tau},
\end{aligned}$$

since P_σ is self-adjoint. ✓

The following result shows that *all* τ-spherical functions on G are associated with principal series.

Proposition B.15. $\Sigma(G, K, \tau, \tau) = \{\Phi_\sigma^\tau(\lambda, \cdot) : \ \sigma \in \widehat{M}_\tau, \ \lambda \in \mathfrak{a}_{\mathbb{C}}^*\}$.

Proof : it is a consequence of a more general result due to Olbrich [Olb94]. Assuming that $\mathbb{D}(G, K, \tau)$ is not necessarily commutative, Olbrich proved that $\Phi_\sigma^\tau(\lambda, \cdot)$ is the unique (normalized) τ-spherical function such that

$$D\, \Phi_\sigma^\tau(\lambda, \cdot)\xi = \mathcal{X}_{\sigma,\lambda}(D)\, \Phi_\sigma^\tau(\lambda, \cdot)\xi \qquad (\forall D \in \mathbb{D}(G, K, \tau), \ \forall \xi \in \mathcal{H}_\tau),$$

where $\mathcal{X}_{\sigma,\lambda}$ is a finite dimensional representation of $\mathbb{D}(G, K, \tau)$ related to $\pi_{\sigma,\lambda}$, and to which one can always restrict. He showed also that the functions $\Phi_\sigma^\tau(\lambda, \cdot)$, when σ and λ vary, exhaust all τ-spherical functions on G if $\mathcal{X}_{\sigma,\lambda}$ is irreducible. In our commutative case, $\mathcal{X}_{\sigma,\lambda}$ is a character of $\mathbb{D}(G, K, \tau)$, and Theorem B.12 shows also that it is actually defined by

$$\mathcal{X}_{\sigma,\lambda}(D) = \tfrac{1}{\dim \tau} \operatorname{tr}\{D\, \Phi_\sigma^\tau(\lambda, \cdot)\}(e), \qquad \forall D \in \mathbb{D}(G, K, \tau).$$

✓

Remarks:

1. The previous result is not optimal, since the action of the Weyl group $W(\mathfrak{g},\mathfrak{a})$ on principal series provides some identifications between the τ-spherical functions $\Phi^\tau_\sigma(\lambda,\cdot)$. For examples, see Section 5.

2. The reason of our restriction to consider only τ-spherical functions associated with a *minimal* principal series representation of G is that they are sufficient, owing to the subrepresentation theorem, to work out the spherical Fourier analysis on $L^2(G,K,\tau,\tau)$.

Let us introduce now a generalized Harish-Chandra c-function, and give some of its properties (see [Wal75] for proofs).

Proposition B.16. *Define, for* $\tau\in\widehat{K}$ *and* $\lambda\in\mathfrak{a}^*_{\mathbb{C}}$,

$$C^\tau(\lambda) := \int_{\overline{N}} d\overline{n}\, e^{-(i\lambda+\rho)(H(\overline{n}))}\tau(\underline{k}(\overline{n})).$$

Then:

(i) $C^\tau(\lambda)\in\operatorname{End}_M\mathcal{H}_\tau$;

(ii) $C^\tau(\lambda)$ *is absolutely convergent for* $\operatorname{Im}\lambda\in\mathfrak{a}^*_-$;

(iii) C^τ *is holomorphic in the domain* $\operatorname{Im}\lambda\in\mathfrak{a}^*_-$ *and has a meromorphic continuation to* $\mathfrak{a}^*_{\mathbb{C}}$;

(iv) $C^\tau(\lambda)$ *is invertible for 'generic'* λ;

(v) *the matrix coefficients of* $C^\tau(\lambda)$ *are expressible in terms of Euler Gamma functions.*

The following result gives the limit behaviour of $\Phi^\tau_\sigma(\lambda,a)$ ($a\in A$), as was done similarly for the vector-valued Poisson transform in [Ven94], [Olb94] and [Yan94]. By '$a\overset{P}{\to}\infty$', we mean that $\alpha(\log a)\to+\infty$ for all $\alpha\in S^+_P$.

Theorem B.17. *For* $\operatorname{Im}\lambda\in\mathfrak{a}^*_-$, *put*

$$c^\tau_\sigma(\lambda) := c_0^{-1}\frac{\dim\tau}{\dim\sigma}C^\tau(\lambda)\circ P_\sigma,$$

where $c_0=\int_{\overline{N}} d\overline{n}\, e^{-2\rho(H(\overline{n}))}$. *Then*

$$\lim_{a\overset{P}{\to}\infty} e^{(\rho-i\lambda)(\log a)}\Phi^\tau_\sigma(-\lambda,a) = c^\tau_\sigma(\lambda)$$

for $\operatorname{Im}\lambda \in \mathfrak{a}^*_-$.

Proof (sketch): We start with the integral formula (B.17). With the standard change-of-variables formula

$$dk = c_0^{-1}\, e^{-2\rho(H(\overline{n}))} d\overline{n},$$

we get

$$\Phi(\lambda, x) = c_{\tau,\sigma}\, c_0^{-1} \int_{\overline{N}} d\overline{n}\, e^{-2\rho(H(\overline{n}))} e^{-(i\lambda+\rho)(H(x\underline{k}(\overline{n}))} \times \tau(\underline{k}(\overline{n})) \circ J^\tau_\sigma(\underline{k}(x\underline{k}(\overline{n}))),$$

where $c_{\tau,\sigma} = \sqrt{\frac{\dim\tau}{\dim\sigma}}$. But since

$$\begin{aligned} H(a\underline{k}(\overline{n})) &= H(a\overline{n}a^{-1}) + \log a - H(\overline{n}), \\ \underline{k}(a\underline{k}(\overline{n})) &= \underline{k}(a\overline{n}a^{-1}), \end{aligned}$$

we have that

$$\Phi(\lambda, a) = c_{\tau,\sigma}\, c_0^{-1}\, e^{-(i\lambda+\rho)(\log a)} \int_{\overline{N}} d\overline{n}\, e^{-(i\lambda+\rho)(H(a\overline{n}a^{-1}))} \times e^{(i\lambda-\rho)(H(\overline{n}))} \tau(\underline{k}(\overline{n})) \circ J^\tau_\sigma(\underline{k}(a\overline{n}a^{-1})).$$

Letting $a \xrightarrow{P} \infty$, we get $a\overline{n}a^{-1} \xrightarrow{P} e$. To exchange limit and integration, one can adapt the classical (but technical) arguments due to Harish-Chandra [HC58a] in the scalar case $\tau = 1$ (see also [Hel84], §IV.6 or [GV88], §4.7). Thus

$$\lim_{a \xrightarrow{P} \infty} e^{(\rho+i\lambda)(\log a)} \Phi^\tau_\sigma(\lambda, a) = c^2_{\tau,\sigma}\, c_0^{-1}\, C^\tau(-\lambda) \circ P_\sigma$$

for $\operatorname{Im}\lambda \in \mathfrak{a}^*_+$, or equivalently

$$\lim_{a \xrightarrow{P} \infty} e^{(\rho-i\lambda)(\log a)} \Phi^\tau_\sigma(-\lambda, a) = c^\tau_\sigma(\lambda)$$

for $\operatorname{Im}\lambda \in \mathfrak{a}^*_-$. ✓

Remarks:

1. The minus sign in $\Phi^\tau_\sigma(-\lambda, a)$ must not be surprising: if we restrict our definition of Φ^τ_σ to $\tau = 1$ (scalar case), then $\Phi^1_1(-\lambda, a) = \varphi_{-\lambda}(a^{-1}) = \varphi_\lambda(a)$, where

$$\varphi_\lambda(x) = \big(\pi_{1,\lambda}(x)1, 1\big)_{L^2(K)} = \int_K dk\, e^{-(i\lambda+\rho)(H(x^{-1}k))}$$

is the standard spherical function, and we recover then the classical behaviour of this function.

2. Proceeding with the study of the function c^τ_σ, one can derive for instance from [War72] a 'Harish-Chandra expansion' for $\Phi^\tau_\sigma(\lambda, a)$ and the relation between c^τ_σ and the Plancherel measure on $L^2(G, K, \tau, \tau)$. There is also an expression of c^τ_σ in terms of Knapp-Stein intertwining operators (see [Ven94] and [Olb94]).
In the case of the bundle we have considered throughout this paper, we see that $c^\tau_\sigma(\lambda)$ is scalar on the one, two or three subspace(s) of $\mathcal{H}_{\tau_{|M}}$, and these scalar components have been completely determined in Section 6 by using the reduction to Jacobi analysis.

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