THE DIFFERENTIAL FORM SPECTRUM OF QUATERNIONIC
HYPERBOLIC SPACES

EMMANUEL PEDON

Abstract. By using harmonic analysis and representation theory, we determine explicitly the $L^2$ spectrum of the Hodge-de Rham Laplacian acting on quaternionic hyperbolic spaces and we show that the unique possible discrete eigenvalue and the lowest continuous eigenvalue can both be realized by some subspace of hypereffective differential forms. Similar results are obtained also for the Bochner Laplacian.

1. Introduction

The problem of computing explicitly (or even of estimating) the $L^2$ spectrum of classical invariant differential operators such as the Dirac operator or the Hodge-de Rham Laplacian acting on a given manifold is generally very delicate. However, when considering particular manifolds such as Riemannian symmetric spaces of noncompact type $G/K$, which possess a rich underlying algebraic structure, one can hope to use $L^2$ harmonic analysis and representation theory of the Lie groups $G$ and $K$ to solve the problem. At least, this point of view has already been successful: for instance the discrete spectrum of the Dirac operator is known in general (see [GS02]) while its continuous spectrum has been calculated in the rank one case (see [CP02]), i.e. when $G/K$ is one of the hyperbolic spaces $H^n(\mathbb{R})$, $H^n(\mathbb{C})$, $H^n(\mathbb{H})$ ($n \geq 2$) or $H^2(\mathbb{O})$.

Let us focus on the case of the Hodge-de Rham Laplacian $\Delta_l$ acting on smooth differential $l$-forms of $G/K$, the operator to which this paper is devoted. Its $L^2$ spectrum is the reunion of a (possibly empty) discrete spectrum and a continuous spectrum, the latter being an subinterval $[\alpha_l, +\infty)$ of $\mathbb{R}_+$. First of all, we have:

**Theorem 1.1** ([Bor85]). The discrete $L^2$ spectrum of $\Delta_l$ on $G/K$ is empty, unless $G$ and $K$ have equal complex rank and $l = \frac{1}{2} \dim_{\mathbb{R}}(G/K)$, in which case it reduces to zero (with infinite multiplicity). Moreover, the space of $L^2$ harmonic forms consists in the sum of all discrete series representations of $G$ having trivial infinitesimal character.

As concerns the continuous spectrum of $\Delta_l$, we miss (so far) a result which is valid for all noncompact symmetric spaces $G/K$, except of course when $l = 0$, in which case it is very well known that

$$\alpha_0 = \|\rho\|^2,$$

where $\rho$ denotes half the sum of positive roots of the pair $(\mathfrak{g}, \mathfrak{a})$, $\mathfrak{g}$ being the Lie algebra of $G$ and $\mathfrak{a}$ a Cartan subspace in $\mathfrak{g}$. For instance, $\|\rho\| = \frac{n-1}{2}$, $n$, $2n + 1$, $11$ when $G/K = H^n(\mathbb{R})$, $H^n(\mathbb{C})$, $H^n(\mathbb{H})$, $H^2(\mathbb{O})$, respectively. Let us recall that $\rho$ is related, as well, to the exponential rate of the volume growth in $G/K$, since indeed

$$\text{vol } B(x, r) \underset{r \to +\infty}{\sim} c e^{2\|\rho\|r}$$

for any point $x \in G/K$.

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The calculation of the bottom \( \alpha_l \) of the continuous spectrum of \( \Delta_l \) for general \( l \) has already been carried out for two particular cases (in what follows, the equality \( \alpha_l = \alpha_{d-l} \), \( d \) being the real dimension of the manifold, reflects Hodge duality):

**Theorem 1.2** (see e.g. [Don81] or [Ped98]). If \( G/K \) is a real hyperbolic space \( H^n(\mathbb{R}) \) \((n \geq 2)\) with constant sectional curvature equal to \(-1\), then \( \alpha_l = \alpha_{n-l} = \left( \frac{n-1}{2} - l \right)^2 \) for \( 0 \leq l \leq \left[ \frac{n}{2} \right] \).

**Theorem 1.3** ([Ped99]). If \( G/K \) is a complex hyperbolic space \( H^n(\mathbb{C}) \) \((n \geq 2)\) with pinched sectional curvature inside \([-4, -1]\), then

\[
\alpha_l = \alpha_{2n-l} = \begin{cases} (n-l)^2 & \text{if } 0 \leq l \leq n-1, \\ 1 & \text{if } l = n. \end{cases}
\]

The main purpose of this paper is to prove a similar result for the last family of noncompact Riemannian symmetric spaces of rank one (the octonionic plane being excepted). Namely, denoting by \( \tau_l \) the \( K \)-representation which defines the bundle of differential \( l \)-forms as a homogeneous vector bundle over \( G/K \), we have:

**Theorem 1.4.** If \( G/K \) is a quaternionic hyperbolic space \( H^n(\mathbb{H}) \) \((n \geq 2)\) with pinched sectional curvature inside \([-4, -1]\), then

\[
\alpha_l = \alpha_{4n-l} = \begin{cases} (2n+1)^2 & \text{if } l = 0, \\ (2n-l)^2 + 8(n-l) & \text{if } 1 \leq l \leq \left[ \frac{4n-1}{6} \right], \\ (2n+1-l)^2 & \text{if } \left[ \frac{4n-1}{6} \right] + 1 \leq l \leq n, \\ (2n-l)^2 & \text{if } n+1 \leq l \leq 2n-1, \\ 1 & \text{if } l = 2n. \end{cases}
\]

Moreover, in each case, the lowest eigenvalue \( \alpha_l \) is realized by some subspace of hyper-effective \( l \)-forms (in the sense of E. Bonan, see Section 2), whose corresponding \( K \)-type occurs with multiplicity one in \( \tau_l \).

As concerns the discrete spectrum, we can make the statement in Theorem 1.1 more precise when \( G/K = H^n(\mathbb{H}) \). Roughly speaking, we are going to prove the following result (see Theorem 4.13 for a complete version; introducing here all the required notation would be tedious):

**Theorem 1.5.** The Hilbert space of \( L^2 \) harmonic \( l \)-forms on \( H^n(\mathbb{H}) \) is trivial unless \( l = 2n \), in which case it consists of the direct sum of the \( n+1 \) discrete series representations of \( G \) which have trivial infinitesimal character, each of them occurring with multiplicity one. Moreover, one (and only one) of these \( n+1 \) summands coincides with the subspace of \( L^2 \) harmonic hypereffective \( 2n \)-forms (its corresponding \( K \)-type occurs with multiplicity one in \( \tau_l \), as well).

Note that the two preceding results say in particular that a part of the spectral information (i.e., the lowest eigenvalue) is completely encoded in the subspace of hypereffective forms, a fact which does not seem obvious to predict from the definitions. However, hypereffective forms are not (in general) the only eigenforms which realize the bottom of the continuous spectrum (see the end of Section 6) and the discrete eigenvalue 0 in middle dimension (by the previous theorem).

The Hodge-de Rham Laplacian \( \Delta_l \) and the Bochner Laplacian \( B_l = \nabla^* \nabla \) are related by a Weitzenböck formula which says that they differ only by a curvature term, which is a zero order differential operator. By calculating this curvature term, we obtain:
Theorem 1.6. Let $\mathcal{B}_l$ denote the Bochner Laplacian on $H^n(\mathbb{H})$.

1. The continuous $L^2$ spectrum of $\mathcal{B}_l$ has the form $[\beta_l, +\infty)$, with

$$
\beta_l = \begin{cases} 
(2n+1)^2 & \text{if } l \text{ is a multiple of } 4, \\
(2n+1)^2 + 3 & \text{if } l \text{ is even but not a multiple of } 4, \\
(2n+1)^2 + 12 & \text{if } l \text{ is odd}.
\end{cases}
$$

2. The discrete $L^2$ spectrum of $\mathcal{B}_l$ is empty, unless $l = 2n$, in which case it consists of the $n + 1$ eigenvalues $8n(n+1) + 4k(k+1-2n)$, with $0 \leq k \leq n$.

We may remark the fact that $\beta_l$ does not ‘fully’ depend on the value of $l$, contrarily to the real case but similarly to the complex case (see Section 7). We shall end our comments with mentioning that the problem of computing the differential form spectrum of general hyperbolic manifolds (i.e., quotients of some hyperbolic space $H^n(F)$ by a discrete and torsion free subgroup of isometries) is much more complicated. However, progress has been made recently in [CP], where lower bounds for the bottom of the spectrum are obtained.

Let us come now to the organization of our article. We have $H^n(\mathbb{H}) = G/K$ with $G = Sp(n, 1)$ and $K = Sp(n) \times Sp(1)$, so we recall in Section 2 some basic facts about this noncompact quaternionic Kähler symmetric space and the related Lie groups and algebras. In Section 3 we make clear the fact that the spectral information we look for is contained in the Plancherel formula for the differential form bundle over $G/K$. Actually, our results are presented in the setting of general hyperbolic spaces $H^n(F)$. This crucial step deserves a particular explanation, so let us elaborate.

The bundle of differential $l$-forms on $H^n(F)$ can be realized as the homogeneous vector bundle associated to the unitary representation $\tau_l = \Lambda^l \text{Ad}_K^*(\sigma)$ of $K$ on $\Lambda^l(T_{\text{cK}}(G/K))_2^\mathbb{C}$. Let $G = KAK$ be a Cartan decomposition of $G$ and let $M$ be the centralizer of $A$ in $K$. Let $\widetilde{M}(\tau_l)$ denote the set of (equivalence classes of) $M$-irreducible factors of the restriction $\tau_l|_M$. To each member $\sigma \in \widetilde{M}(\tau_l)$ is attached a nonnegative real number $c(\sigma)$ called the Casimir value of $\sigma$ (see formula (3.7)) and we denote by $\sigma_{\text{max}}$ one of the elements of $\widetilde{M}(\tau_l)$ such that $c(\sigma_{\text{max}}) \geq c(\sigma)$ for all $\sigma \in \widetilde{M}(\tau_l)$. By using Harish-Chandra’s Plancherel formula for rank one Lie groups, we prove first that the bottom $\alpha_l$ of the continuous $L^2$ spectrum of $\Delta_l$ equals

$$
\alpha_l = \rho^2 - c(\sigma_{\text{max}}).
$$

Actually this result is just the generalization of (1.1) to differential forms in the rank one case, taking into account that $\rho$ can be identified with its norm in this particular setting. On the other hand, we slightly improve Theorem 1.1 by identifying each occurring discrete series representation with the harmonic part of the subspace of $L^2$ forms corresponding to a particular $K$-type decomposing $\tau_l$.

Thus, to calculate both the discrete and continuous spectrum of $\Delta_l$, we need the full decomposition of $\tau_l$ into $K$-irreducible constituents, for any $l$. This quite fastidious task is carried out in Section 4, essentially by using results of P. Möseneder Frajria [Mö83]. Since the decomposition we give concerns actually all quaternionic Kähler manifolds (of which $H^n(\mathbb{H})$ represents the noncompact prototype), our results recover and generalize the ones due to E. Bonan [Bon95b] and A. Swann [Swa89] for $l \leq 5$. Then, by identifying, among all irreducible summands of $\tau_l$, those which correspond to hypereffective forms, we prove a result which implies Theorem 1.5 (see Theorem 4.13).
Next, we go back to the calculation of \( \alpha_l \). In view of (1.2), we need to describe the set \( \hat{M}(\tau_l) \), i.e. to decompose each \( K \)-irreducible factors of \( \tau_l \) into \( M \)-irreducible components. We do this in Section 5 by using the branching laws due to M. W. Baldoni Silva [Bal79].

Lastly, Section 6 is devoted to the explicit calculation of \( c(\sigma_{\text{max}}) \) in order to achieve the proof of Theorem 1.4, while Section 7 deals with the proof of Theorem 1.6.

2. Quaternionic hyperbolic spaces and related Lie theory

Let \( n \geq 2 \) be an integer. In the sequel, any vector space on \( \mathbb{H} \), and in particular \( \mathbb{H}^{n+1} \), will be considered as a right vector space. Define the Lorentz form \( \ell \) on \( \mathbb{H}^{n+1} \) by

\[
\ell(x, y) = \overline{y_1}x_1 + \cdots + \overline{y_{n+1}}x_{n+1},
\]

where \( \overline{y} \) is the quaternionic conjugate of \( y \). The quaternionic hyperbolic space of dimension \( n \) is the quotient

\[
H^n(\mathbb{H}) = \{ x \in \mathbb{H}^{n+1} : \ell(x, x) < 0 \}/H^n.
\]

Let \( G \) be the subgroup of \( GL(n+1, \mathbb{H}) \) preserving \( \ell \), and let \( K \) be the isotropy subgroup of the origin \( o = (0, \ldots, 0, 1)H^n \) of \( H^n(\mathbb{H}) \). Then \( G = Sp(n, 1) \) belongs to the class of connected noncompact semisimple real Lie groups with finite centre, \( K \simeq Sp(n) \times Sp(1) \) is a maximal compact subgroup of \( G \), and the quaternionic hyperbolic space \( H^n(\mathbb{H}) \) can be realized as the homogeneous manifold \( G/K \) and equipped with a \( G \)-invariant Riemannian metric \( g \) (which we shall normalize at the end of this section). More precisely, \( H^n(\mathbb{H}) \) is a noncompact Riemannian symmetric space of rank one and of real dimension \( 4n \) (see for instance [Hel78], Table V p. 518).

Let \( \mathfrak{g} = sp(n, 1) \) and \( \mathfrak{k} = sp(n) \oplus sp(1) \) be the Lie algebras of \( G \) and \( K \), respectively, and write

\[
(2.1) \quad \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}
\]

for the Cartan decomposition of \( \mathfrak{g} \). The subspace \( \mathfrak{p} \) is thus identified with the tangent space \( T_o(G/K) \simeq H^n \) of \( H^n(\mathbb{H}) = G/K \) at the origin \( o = eK \), and this isomorphism exhibits actually a \( K \)-equivalence between the isotropy representation at \( o \) and the adjoint representation

\[
(2.2) \quad \text{Ad} : k \mapsto \text{Ad}_G(k)|_{\mathfrak{p}}.
\]

On the other hand, \( H^n(\mathbb{H}) \) has holonomy group \( Sp(n)Sp(1) \subset SO(4n) \) (isomorphic to \( K/Z_2 \)) or, in other words, \( H^n(\mathbb{H}) \) is a quaternionic Kähler manifold (see e.g. [Bes87], Ch. 14 or [Sal89], §9). In particular, as a quaternion-Hermitian manifold \( H^n(\mathbb{H}) \) is equipped with three local 2-forms \( \omega_1, \omega_2, \omega_3 \) associated with a local basis \((I_1, I_2, I_3)\) of almost complex structures which behave as imaginary quaternions \((I_1I_2 = -I_2I_1 = I_3)\).

Namely, for \( k = 1, 2, 3 \):

\[
(2.3) \quad L_k(f) = \omega_k \wedge f, \quad L^*_k(f) = \ast(\omega_k \wedge \ast(f)),
\]

i.e. \( L_k^* \) is the adjoint of \( L_k \). Following E. Bonan [Bon95a], we say that a differential form \( f \) is hypereffective if \( L_k^*(f) = 0 \) for \( k = 1, 2, 3 \). Note that hypereffective forms are automatically effective, i.e. they verify \( L^*(f) = 0 \), where \( L^* \) is defined by a formula similar to (2.3) as the adjoint of \( L \), the left exterior multiplication by the (global, parallel, non degenerate) fundamental 4-form

\[
\Omega = \omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 + \omega_3 \wedge \omega_3.
\]
Next we need to recall some structure of the Lie algebras related to $H^n(\mathbb{H})$. By using the classical identification between $\mathbb{H}$ and $\mathbb{C}^2$, the following descriptions are standard (see e.g. [Hel78], Ch. X):

\[ g = \left\{ \begin{pmatrix} A & c & B & d \\ t\bar{c} & u & t\bar{d} & v \\ -B & \bar{d} & \bar{A} & -\bar{c} \\ t\bar{d} & -\bar{v} & -t\bar{c} & \bar{u} \end{pmatrix} \mid A \in u(n), B \in \text{Sym}(n, \mathbb{C}), \quad c, d \in \mathbb{C}^n, u \in u(1), v \in \mathbb{C} \right\}, \]

\[ \mathfrak{k} = \left\{ \begin{pmatrix} A & 0 & B & 0 \\ 0 & u & 0 & v \\ -B & 0 & \bar{A} & 0 \\ 0 & -\bar{v} & 0 & \bar{u} \end{pmatrix} \mid A \in u(n), B \in \text{Sym}(n, \mathbb{C}), \quad u \in u(1), v \in \mathbb{C} \right\}, \]

\[ \mathfrak{p} = \left\{ \begin{pmatrix} 0 & c & 0 & d \\ t\bar{c} & 0 & t\bar{d} & 0 \\ 0 & d & 0 & -\bar{c} \\ t\bar{d} & 0 & -t\bar{c} & 0 \end{pmatrix} \mid c, d \in \mathbb{C}^n \right\}. \]  

(2.4)

Let $\mathfrak{a} = RH_0$, where $H_0 \in \mathfrak{p}$ has $t^c = (1,0,\ldots,0)$ and $t^d = (0,\ldots,0)$ in the parametrization above. Then $\mathfrak{a}$ is a Cartan (i.e., a maximal abelian) subspace in $\mathfrak{p}$. Let $\mathfrak{m}$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$, then $\mathfrak{m} \simeq \mathfrak{sp}(n-1) \oplus \mathfrak{sp}(1)$ and, more specifically:

\[ \mathfrak{m} = \left\{ \begin{pmatrix} u & 0 & 0 & -v & 0 & 0 \\ 0 & A & 0 & 0 & B & 0 \\ 0 & 0 & u & 0 & 0 & v \\ \bar{v} & 0 & 0 & \bar{u} & 0 & 0 \\ 0 & -\bar{B} & 0 & 0 & \bar{A} & 0 \\ 0 & 0 & -\bar{v} & 0 & 0 & \bar{u} \end{pmatrix} \mid A \in u(n-1), B \in \text{Sym}(n-1, \mathbb{C}), \quad u \in u(1), v \in \mathbb{C} \right\}. \]

Since $g$ and $\mathfrak{k}$ have equal complex rank, let $\mathfrak{h}$ be the common Cartan subalgebra constituted with diagonal elements. Likewise, let $\mathfrak{t}$ be the Cartan subalgebra of $\mathfrak{m}$ constituted with diagonal elements. For $1 \leq j \leq n + 1$, let $E_j \in \mathfrak{h}_\mathbb{C}$ denote the matrix defined by $(E_j)_{ii} = 1$ if $i = j$, $(E_j)_{ii} = -1$ if $i = n + 1 + j$ and $(E_j)_{ii} = 0$ else. Then $(E_j)$ is a basis of $\mathfrak{h}_\mathbb{C}$. Denote by $(\varepsilon_j)$ the corresponding dual basis of $\mathfrak{h}_\mathbb{C}^\ast$ (for convenience, we shall keep the same notation for the restriction of $\varepsilon_j$ to $\mathfrak{t}_\mathbb{C}$). We have then the following standard descriptions of root systems:

\[ \Delta_{\mathfrak{g}} = \{ \pm \varepsilon_i \pm \varepsilon_j, \ 1 \leq i < j \leq n + 1 \} \cup \{ \pm 2\varepsilon_k, \ 1 \leq k \leq n + 1 \}, \]

(2.5)

\[ \Delta_{\mathfrak{f}} = \{ \pm \varepsilon_i \pm \varepsilon_j, \ 1 \leq i < j \leq n \} \cup \{ \pm 2\varepsilon_k, \ 1 \leq k \leq n + 1 \}, \]

(2.6)

\[ \Delta_{\mathfrak{m}} = \{ \varepsilon_i \pm \varepsilon_{n+1}; \ \pm \varepsilon_i \pm \varepsilon_j, \ 2 \leq i < j \leq n \} \cup \{ \pm 2\varepsilon_k, \ 2 \leq k \leq n \}. \]

With the usual ordering, they admit as positive subsystems:

\[ \Delta^+_{\mathfrak{g}} = \{ \varepsilon_i \pm \varepsilon_j, \ 1 \leq i < j \leq n + 1 \} \cup \{ 2\varepsilon_k, \ 1 \leq k \leq n + 1 \}, \]

(2.7)

\[ \Delta^+_{\mathfrak{f}} = \{ \varepsilon_i \pm \varepsilon_j, \ 1 \leq i < j \leq n \} \cup \{ 2\varepsilon_k, \ 1 \leq k \leq n + 1 \}, \]

(2.8)

\[ \Delta^+_{\mathfrak{m}} = \{ \varepsilon_i \pm \varepsilon_{n+1}; \ \varepsilon_i \pm \varepsilon_j, \ 2 \leq i < j \leq n \} \cup \{ 2\varepsilon_k, \ 2 \leq k \leq n \}, \]
and the corresponding half-sums of positive roots are

\begin{align}
\delta_g &= \sum_{j=1}^{n+1} (n + 2 - j) \varepsilon_j, \\
\delta_k &= \sum_{j=1}^{n} (n + 1 - j) \varepsilon_j + \varepsilon_{n+1}, \\
\delta_m &= \sum_{j=2}^{n} (n + 1 - j) \varepsilon_j + \frac{1}{2}(\varepsilon_1 + \varepsilon_{n+1}).
\end{align}

As is well-known, the set of (equivalence classes of) irreducible finite dimensional representations of the connected compact group Lie $K$ (resp. $M$) is in one-to-one correspondence with the set $D_K$ (resp. $D_M$) of dominant analytically integral weights. By Lemmas 5.2 & 5.3 in [Bal79], we have

\begin{align}
D_K &= \left\{ \sum_{j=1}^{n+1} a_j \varepsilon_j, \ a_j \in \mathbb{N} \text{ for all } j, \ a_1 \geq \cdots \geq a_n \right\}, \\
D_M &= \left\{ b_0(\varepsilon_1 + \varepsilon_{n+1}) + \sum_{j=2}^{n} b_j \varepsilon_j, \ 2b_0 \in \mathbb{N}, \ b_j \in \mathbb{N} \text{ for all } j \geq 2, \ b_2 \geq \cdots \geq b_n \right\}
\end{align}

Next, let $R(\mathfrak{g}, \mathfrak{a})$ be the restricted root system of the pair $(\mathfrak{g}, \mathfrak{a})$, with positive subsystem $R^+(\mathfrak{g}, \mathfrak{a})$ corresponding to the positive Weyl chamber $\mathfrak{a}_+ \simeq (0, +\infty)$ in $\mathfrak{a} \simeq \mathbb{R}$. Then there exists a linear functional $\alpha \in \mathfrak{a}^*$ such that $R(\mathfrak{g}, \mathfrak{a}) = \{ \pm \alpha, \pm 2\alpha \}$ and $R^+(\mathfrak{g}, \mathfrak{a}) = \{ \alpha, 2\alpha \}$. As usual, we write $\mathfrak{n}$ for the direct sum of positive root subspaces, i.e. $\mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$, so that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ is an Iwasawa decomposition for $\mathfrak{g}$. We let also $\rho = \frac{1}{2}(m_\alpha \alpha + m_{2\alpha} 2\alpha)$, where $m_\alpha = \dim_{\mathbb{R}} \mathfrak{g}_\alpha = 4(n-1)$ and $m_{2\alpha} = \dim_{\mathbb{R}} \mathfrak{g}_{2\alpha} = 3$. In the sequel, we shall use systematically the identification

\begin{align}
\mathfrak{a}^* &\simeq \mathbb{R}, \\
\lambda \alpha &\mapsto \lambda.
\end{align}

In particular, we shall view $\rho$ as a real number, namely

\begin{align}
\rho = 2n + 1.
\end{align}

Now, we define a symmetric bilinear form on $\mathfrak{g}$ by

\begin{align}
\langle X, Y \rangle = \frac{1}{B(H_0, H_0)} B(X, Y) = \frac{1}{4} \text{tr}_C(XY),
\end{align}

where $B$ is the Killing form on $\mathfrak{g}$, given by

\begin{align}
B(X, Y) = 2(n+2) \text{tr}_C(XY).
\end{align}

Hence $\langle \cdot, \cdot \rangle$ is positive definite on $\mathfrak{p}$, negative definite on $\mathfrak{k}$ and we have

\begin{align}
\langle \mathfrak{p}, \mathfrak{p} \rangle = 0.
\end{align}

Note that our normalization is made so that the $\text{Ad}_G(K)$-invariant scalar product on $\mathfrak{p} \cong T_0(G/K)$ defined by the restriction of $\langle \cdot, \cdot \rangle$ induces a $G$-invariant Riemannian metric $g$ on $H^n(\mathbb{H})$ which has pinched sectional curvature inside the interval $[-4, -1]$. 
The bilinear form \((2.16)\) induces a bilinear form on \(h^*_C\) (and on \(m^*_C\)) as well, and it is easy to observe that
\[
\langle \varepsilon_i, \varepsilon_j \rangle = 2\delta_{ij}.
\]

3. Harmonic analysis for differential forms on a noncompact Riemannian symmetric space of rank one

In this section, we shall illustrate a very basic principle of harmonic analysis, namely: a Plancherel formula for a given Hilbert space reflects the spectral decomposition of some ‘natural’ differential operator acting on this space. Of course, we intend to treat here the case of the Hodge-de Rham Laplacian \(\Delta_l\) acting on differential \(l\)-forms of \(G/K\). Actually, in all this section, \(G/K\) will denote any noncompact Riemannian symmetric space of rank one, i.e. any member of the list

\[
\begin{align*}
H^n(\mathbb{R}) &= SO_e(n,1)/SO(n), \\
H^n(\mathbb{C}) &= SU(n,1)/SU(n) \times U(1), \\
H^n(\mathbb{H}) &= Sp(n,1)/Sp(n) \times Sp(1), \\
H^2(\mathbb{O}) &= F_4(-20)/Spin(9),
\end{align*}
\]

since no extra effort is required to state the results at this level of generality (it is understood that the Lie algebras \(a, m, n\) introduced in Section 2 can be defined similarly in each of the other cases). As it will be indicated, parts of our discussion will remain valid in even more general situations.

3.1. The differential form bundle over \(G/K\). Let \((\tau, V_\tau)\) be a unitary finite dimensional representation of the group \(K\) (not necessarily irreducible). It is standard (see e.g. [Wal73], §5.2) that the space of sections of the \(G\)-homogeneous vector bundle \(E_\tau = G \times_K V_\tau\) can be identified with the space
\[
\Gamma(G, \tau) = \{ f : G \to V_\tau, \ f(xk) = \tau(k)^{-1}f(x), \forall x \in G, \forall k \in K \}
\]
of functions of (right) type \(\tau\) on \(G\). We define also the subspaces
\[
C^\infty(G, \tau) = \Gamma(G, \tau) \cap C^\infty(G, V_\tau), \quad \text{and} \quad L^2(G, \tau) = \Gamma(G, \tau) \cap L^2(G, V_\tau)
\]
of \(\Gamma(G, \tau)\) which correspond to \(C^\infty\) and \(L^2\) sections of \(E_\tau\), respectively. Note that \(L^2(G, \tau)\) is the Hilbert space associated with the unitary induced representation \(\text{Ind}^G_K(\tau)\) of \(G\), the action being given by left translations.

For \(0 \leq l \leq d\) (here \(d = \text{dim}_\mathbb{C}(G/K)\)), let \(\tau_l\) denote the \(l\)-th exterior product of the complexified coadjoint representation \(Ad^*_C\) of \(K\) on \(p^*_C\) (see (2.2)). Then \(\tau_l\) is a unitary representation of \(K\) on \(V_{\tau_l} = \wedge^l p^*_C\) and the corresponding homogeneous bundle \(E_{\tau_l}\) is the bundle of differential forms of degree \(l\) on \(G/K\).

In general, the representation \(\tau_l\) is not \(K\)-irreducible and decomposes as a finite direct sum of \(K\)-types:
\[
\tau_l = \bigoplus_{\tau \in \hat{K}} m(\tau, \eta)\tau,
\]
where \(m(\tau, \eta) \geq 0\) is the multiplicity of \(\tau\) in \(\tau_l\) (as usual, \(\hat{K}\) stands for the unitary dual of the Lie group \(K\)). Let us set
\[
\hat{K}(\tau_l) = \{ \tau \in \hat{K}, m(\tau, \eta_l) > 0 \},
\]
so that (3.1) induces the decomposition

\begin{equation}
L^2(G, \tau) = \bigoplus_{\tau \in \hat{K}(\tau)} L^2(G, \tau) \otimes \mathbb{C}^{m(\tau, \pi)},
\end{equation}

as well as its analogue for $C^\infty(G, \tau)$.

Let $\tau \in \hat{K}$. The Plancherel formula for the space $L^2(G, \tau)$ consists in the diagonalization of the corresponding unitary representation $\text{Ind}^G_K(\pi)$ of $G$. First, we remark that

\[ L^2(G, \tau) \simeq \{ L^2(G) \otimes V_\tau \}^K, \]

where the upper index $K$ means that we take the subspace of $K$-invariant vectors for the right action of $K$ on $L^2(G)$. According to Harish-Chandra’s famous Plancherel Theorem for $L^2(G)$ (see e.g. [Kna86]), the space $L^2(G, \tau)$ splits then into the direct sum of a continuous part $L^2_c(G, \tau)$ and of a discrete part $L^2_d(G, \tau)$. Consequently, by summing over the set $\hat{K}(\tau)$ we get a corresponding decomposition of our bundle of $L^2$ differential $l$-forms

\[ L^2(G, \tau) = L^2_c(G, \tau) \oplus L^2_d(G, \tau). \]

In order to make both terms in this decomposition become more explicit and to see how the spectrum of $\Delta_\tau$ can be derived from their expression, we shall now consider them separately.

### 3.2. The continuous part of the Plancherel formula for $L^2(G, \tau)$

First of all, we need to recall some basic facts which concern principal series representations of $G$ (see e.g. [Kna86] or [Wal73]). Denote by $A$, $M$ and $N$ the analytic Lie subgroups of $G$ that correspond respectively to the Lie algebras $a$, $m$ and $n$. For $t \in \mathbb{R}$, we set $a_t = \exp(tH_0)$, so that

\[ A = \{ a_t, t \in \mathbb{R} \}. \]

Let $P = MAN$ be the standard minimal parabolic subgroup of $G$. For $\sigma \in \hat{M}$ and $\lambda \in a_\mathbb{C}^* \simeq \mathbb{C}$, the principal series representation $\pi_{\sigma, \lambda}$ of $G$ is the induced representation

\[ \pi_{\sigma, \lambda} = \text{Ind}_{\hat{P}^\perp}^G(\sigma \otimes e^{\lambda} \otimes 1) \]

with corresponding space

\[ H^{\infty}_{\sigma, \lambda} = \{ f \in C^{\infty}(G, V_\sigma), f(xma_tn) = e^{-(\lambda+\rho)t}\sigma(m)^{-1}f(x), \forall x \in G, \forall ma_tn \in P \}. \]

This $G$-action is given by left translations: $\pi_{\sigma, \lambda}(g)f(x) = f(g^{-1}x)$. Moreover, if $H_{\sigma, \lambda}$ denotes the Hilbert completion of $H^{\infty}_{\sigma, \lambda}$ with respect to the norm $\| f \| = \| f \|_{L^2(K)}$, then $\pi_{\sigma, \lambda}$ extends to a continuous representation of $G$ on $H_{\sigma, \lambda}$. When $\lambda \in \mathbb{R}$, the principal series representation $\pi_{\sigma, \lambda}$ is unitary, in which case it is also irreducible, except maybe for $\lambda = 0$.

Now, let us consider any irreducible unitary representation $\tau \in \hat{K}$. When restricted to the subgroup $M$ of $K$, it is generally no more irreducible, and splits into a finite direct sum

\[ \tau|_M = \bigoplus_{\sigma \in \hat{M}} m(\sigma, \tau)\sigma, \]

where $m(\sigma, \tau) \geq 0$ is the multiplicity of $\sigma$ in $\tau|_M$. Let us define then

\begin{equation}
\hat{M}(\tau) = \{ \sigma \in \hat{M}, m(\sigma, \tau) > 0 \}.
\end{equation}
The continuous part of the Plancherel formula for the space $L^2(G, \tau)$ takes the following form (see e.g. [Ped99], §3, for details):

$$L^2_c(G, \tau) \simeq \bigoplus_{\tau \in \hat{K}(\tau)} \bigoplus_{\sigma \in \tilde{M}(\tau)} \int_{a^*_+}^{\oplus} d\lambda \rho_\sigma(\lambda) H_{\sigma,\lambda} \otimes \text{Hom}_K(\mathcal{H}_{\sigma,\lambda}, V_{\tau})$$

In this formula, $d\lambda$ is the Lebesgue measure on $a^*_+ \simeq (0, +\infty)$, $\rho_\sigma(\lambda)$ is the Plancherel density associated with $\sigma$ and $\text{Hom}_K(\mathcal{H}_{\sigma,\lambda}, V_{\tau})$ is the vector space of $K$-intertwining operators from $\mathcal{H}_{\sigma,\lambda}$ to $V_{\tau}$, on which $G$ acts trivially. This space is non trivial (since $\sigma \in \hat{M}(\tau)$) but finite dimensional (since every irreducible unitary representation of $G$ is admissible).

By combining formulas (3.2) and (3.4) we get immediately the following result.

**Theorem 3.1.** The continuous part of the Plancherel formula for $L^2(G, \tau)$ is given by:

$$L^2_c(G, \tau) \simeq \bigoplus_{\tau \in \hat{K}(\tau)} \bigoplus_{\sigma \in \tilde{M}(\tau)} \int_{a^*_+}^{\oplus} d\lambda \rho_\sigma(\lambda) H_{\sigma,\lambda} \otimes \text{Hom}_K(\mathcal{H}_{\sigma,\lambda}, V_{\tau}) \otimes \mathbb{C}^{m(\tau, \gamma)}.$$

Now we explain how this formula leads to the determination of $\alpha_l$. The Hodge-de Rham Laplacian $\Delta_l = dd^* + d^*d$ acts on $C^\infty$ differential $l$-forms on $G/K$, i.e. on members of the space $C^\infty(G, \tau)$. Actually, this operator is realized by the action of the Casimir element $\Omega_\mathfrak{g}$ of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. More precisely, keeping notation (2.16), let $(Z_i)$ be any basis for $\mathfrak{g}$ and $(Z^i)$ the corresponding basis of $\mathfrak{g}$ such that $\langle Z_i, Z^j \rangle = \delta_{ij}$. The Casimir operator can be written as

$$\Omega_\mathfrak{g} = \sum_i Z_i Z^i.$$  

We can regard $\Omega_\mathfrak{g}$ as a $G$-invariant differential operator acting on $C^\infty(G, \eta)$, and have then the well-known identification (Kuga’s formula, see [BW00], Theorem II.2.5)

$$\Delta_l \equiv -\Omega_\mathfrak{g}.$$  

In order to investigate the continuous $L^2$ spectrum of $\Delta_l$, it is thus enough to consider the action of the Casimir operator $\Omega_\mathfrak{g}$ on the right-hand side of the Plancherel formula given in Theorem 3.1 and, specifically, on each ‘elementary component’ $H_{\sigma,\lambda} \otimes \text{Hom}_K(\mathcal{H}_{\sigma,\lambda}, V_{\tau})$.

The action of $\Omega_\mathfrak{g}$ on $\text{Hom}_K(\mathcal{H}_{\sigma,\lambda}, V_{\tau})$ being trivial, the problem reduces to study its effect on $H_{\sigma,\lambda}$, and even on $H_{\sigma,\lambda}^\infty$ by density. But since $\Omega_\mathfrak{g}$ is a central element in the enveloping algebra of $\mathfrak{g}$, it acts on the irreducible admissible representation $H_{\sigma,\lambda}^\infty$ by a scalar $\omega_{\sigma,\lambda}$. Let us elaborate: let $\mu_\sigma$ be the highest weight of $\sigma \in \hat{M}$ and recall that $\delta_m$ was defined in (2.11). Then $\sigma(\Omega_m) = -c(\sigma) \text{Id}$, where the Casimir value of $\sigma$ is given by

$$c(\sigma) = \langle \mu_\sigma, \mu_\sigma + 2\delta_m \rangle \geq 0.$$

Using for instance [Kna86], Proposition 8.22 and Lemma 12.28, one easily checks that

$$\Omega_\mathfrak{g} = - (\lambda^2 + \rho^2 - c(\sigma)) \text{Id} \text{ on } H_{\sigma,\lambda}^\infty.$$

Thus (3.8), (3.6) and Theorem 3.1 show that the action of $\Delta_l$ on $L^2(G, \tau)$ is diagonal, and this allows us to calculate the continuous $L^2$ spectrum of $\Delta_l$: set

$$\hat{M}(\tau) = \bigcup_{\tau \in \hat{K}(\tau)} \hat{M}(\tau),$$
and denote by $\sigma_{\max}$ one of the (possibly many) elements of $\widehat{M}(\tau_l)$ such that $c(\sigma_{\max}) \geq c(\sigma)$ for any $\sigma \in \widehat{M}(\tau_l)$. Our discussion implies immediately the following result.

**Theorem 3.2.** The continuous $L^2$ spectrum of the Hodge-de Rham Laplacian $\Delta_l$ is $[\alpha_l, +\infty)$, with $\alpha_l = \rho^2 - c(\sigma_{\max})$.

### 3.3. The discrete part of the Plancherel formula for $L^2(G, \tau_l)$

In this subsection, we remove the assumption on the rank of $G/K$, since our discussion will be valid for any noncompact Riemannian symmetric space.

As a consequence of Harish-Chandra’s Plancherel Theorem, the discrete part of $L^2(G, \tau_l)$ reads

$$L^2_d(G, \tau_l) = \bigoplus_{\pi \in \widehat{G}_d} d_{\pi} H_{\pi} \otimes \text{Hom}_K(H_{\pi}, V_{\tau_l}),$$

where $\widehat{G}_d$ denotes the set of (equivalence classes of) discrete series representations of $G$, i.e. of irreducible unitary representations having $L^2$ matrix coefficients, and $d_{\pi}$ denotes the formal degree of such a representation $(\pi, H_{\pi})$ (see e.g. [Kna86], §IX.3). We shall assume that $G$ and $K$ have equal complex rank, since the set $\widehat{G}_d$ is empty otherwise (by another famous result of Harish-Chandra). When $G/K = H^n(\mathbb{F})$ is of rank one, we always have $\text{rk}(G) = \text{rk}(K)$, unless $\mathbb{F} = \mathbb{R}$ and $n$ is odd.

In order to go farther in the exploration of the previous formula, we need to generalize the notations of Section 2 to our situation. Let $\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{g}$ be a Cartan subalgebra, let $\Delta_l \subset \Delta_\mathfrak{g}$ and $W_l \subset W_\mathfrak{g}$ be the corresponding root systems and Weyl groups. Once a positive subsystem $\Delta_+^l$ is fixed in $\Delta_l$, there are exactly $m = |W_\mathfrak{g}|/|W_l|$ positive subsystems in $\Delta_\mathfrak{g}$ whose intersection with $\Delta_\mathfrak{k}$ coincides with $\Delta_\mathfrak{k}^+$. Let $\Delta_\mathfrak{g}^+$ be one of them and let $i\mathfrak{h}_\mathfrak{g}^+$ denote the corresponding positive $G$-Weyl chamber in $i\mathfrak{h}$. Then any positive subsystem in $\Delta_\mathfrak{g}$ can be written as $w_j \cdot \Delta_\mathfrak{g}^+$, where $w_1, \ldots, w_m$ are distinguished representatives of $W_\mathfrak{k} \backslash W_\mathfrak{g}$ in $W_\mathfrak{g}$. Let $\delta_\mathfrak{g}$ and $\delta_\mathfrak{k}$ denote the half-sums of roots in $\Delta_\mathfrak{g}^+$ and $\Delta_\mathfrak{k}^+$, respectively. As is well-known (see e.g. [Kna86], §IX.7), discrete series representations are, up to equivalence, uniquely determined by their Harish-Chandra parameter $w_k \cdot \Lambda$, where $\Lambda \in (i\mathfrak{h}_\mathfrak{g}^+)^*$ is such that $\Lambda + \delta_\mathfrak{g}$ is analytically integral. For $1 \leq k \leq m$, let $(\pi_k, H_k)$ denote the discrete series representation of $G$ whose Harish-Chandra parameter is $w_k \cdot \delta_\mathfrak{g}$. In other words, these are exactly the discrete series representation of $G$ which have trivial infinitesimal character.

The following refinement of Theorem 1.1 was proved in [Ped97].

**Theorem 3.3.** Set $d = \text{dim}_\mathbb{R}(G/K)$ and let $0 \leq l \leq d$.

1. If $l \neq \frac{d}{2}$, then $L^2_d(G, \tau_l) = \{0\}$.
2. If $l = \frac{d}{2}$, $L^2_d(G, \tau_l)$ consists of the sum $\bigoplus_{k=1}^m H_k$, i.e. of all discrete series of $G$ with trivial infinitesimal character, each of them occurring with multiplicity one. Moreover, each $H_k$ is realized as the null space for the Casimir operator $\Omega_{\mathfrak{g}}$ acting on $L^2(G, \tau_{\lambda_k})$, where $\tau_{\lambda_k}$ is the multiplicity free subrepresentation of $\tau_{\mathfrak{g}}^+$ with highest weight $\lambda_k = w_k \cdot 2\delta_\mathfrak{g} - 2\delta_\mathfrak{k}$ and is the minimal $K$-type of $\pi_k$.

Because of Kuga’s formula (3.6), this result yields a description of the Hilbert space of $L^2$ harmonic forms on $G/K$, and in particular on $H^n(\mathbb{X})$. We shall come back on this subject at the end of next section (see Theorem 4.13).

**Remark 3.4.** Similar considerations lead to the calculation of the spectrum of the Dirac operator. See [GS02] and [CP02].
4. The $K$-decomposition of $\tau_1$

Now we turn back to our particular case $G/K = H^n(\mathbb{H})$. As explained in the previous section, we need to decompose into irreducibles the $K$-representation $\tau_1 = \wedge^l \text{Ad}_C$ on $V_{\tau_1} = \wedge^l p_C^*$, for $0 \leq l \leq 4n$ (actually, considering $0 \leq l \leq 2n$ is enough by Hodge duality).

This problem was solved by P. Möseneder Frajria in his Master’s Thesis [Mö83], although the decomposition was not stated in a fully explicit way. Since the results are unpublished, we think it is worth while to include most of them in this paper, together with the proofs when these are non trivial. Then, we shall go a little farther than P. Möseneder Frajria’s results to make the $K$-decomposition of $\tau_1$ totally explicit.

In the sequel, we shall handle finite dimensional representations of the Lie group $K = \text{Sp}(n) \times \text{Sp}(1)$ and of its subgroups $K^0 = \text{Sp}(n) \times \{1\}$ and $K^1 = \{1\} \times \text{Sp}(1)$. Since all these groups are compact, connected and simply connected, dominant analytically integral weights, so that we have a one-to-one correspondence between the irreducible finite dimensional representations of the Lie groups and the irreducible finite dimensional representations of the corresponding (complexified) Lie algebras. This identification will be used systematically thereafter.

We let $\mathfrak{k}^0$ and $\mathfrak{k}^1$ denote the respective Lie algebras of $K^0$ and $K^1$, so that

$$\mathfrak{k}_C = \mathfrak{k}_C^0 \oplus \mathfrak{k}_C^1, \quad \text{with } \mathfrak{k}_C^0 = \text{sp}(n, \mathbb{C}) \text{ and } \mathfrak{k}_C^1 = \text{sp}(1, \mathbb{C}) \simeq \mathfrak{sl}(2, \mathbb{C}).$$

Let us remind that every representation of $\mathfrak{sp}(m, \mathbb{C})$, hence of $\mathfrak{k}_C^0, \mathfrak{k}_C^1, \mathfrak{k}_C$, is self contragredient (for the Weyl group contains $-\text{Id}$). This will always be understood in the remaining of this section.

As a consequence of the preceding discussion, our task reduces to decompose the representation $\wedge^l \text{ad}_C$ of $\mathfrak{k}_C$ on $\wedge^l p_C$, which will also be denoted by $\tau_1$.

Let $\varphi_m$ be the standard complex representation of $\mathfrak{sp}(m, \mathbb{C})$ on $\mathbb{C}^{2m}$. Clearly, we have an equivalence

$$\tau_1 \sim \wedge^l (\varphi_n \otimes \varphi_1)$$

as representations of $\mathfrak{k}_C = \mathfrak{k}_C^0 \oplus \mathfrak{k}_C^1$. Thus the decomposition of $\tau_1$ can be obtained in theory by applying the Schur functors machinery (see e.g. [FH91], Lecture 6). However, we are convinced that P. Möseneder Frajria’s approach [Mö83] is more elegant and easier to develop. His main idea was to observe that the problem of decomposing $\tau_1$ reduces to the one of decomposing some particular representation of $\mathfrak{k}_C^0 = \mathfrak{sp}(n, \mathbb{C})$. Let us elaborate.

The very first step consists in decomposing $\wedge^l p_C$ as a $\mathfrak{k}_C^0$-module. Consistently with (2.4), we parametrize $p_C \simeq \mathbb{C}^{2n}$ by elements

$$A(c, c', d, d') = \begin{pmatrix} 0 & c & 0 & d \\ \ell c' & 0 & \ell d & 0 \\ 0 & d' & 0 & -c' \\ \ell d' & 0 & -\ell c & 0 \end{pmatrix}, \quad \text{with } c, c', d, d' \in \mathbb{C}^n,$$

so that the map

$$T : A(c, c', d, d') \mapsto \begin{pmatrix} c \\ d' \end{pmatrix} + (\ell c' \ell d)$$

induces an identification

$$p_C \simeq \mathbb{C}^{2n} \oplus (\mathbb{C}^{2n})^*$$

(4.2)
that we shall use systematically in the sequel. In particular, we have an isomorphism

\[(4.3) \quad \wedge^l p_C \simeq \bigoplus_{p+q=l} \wedge^{p,q},\]

where we have set

\[\wedge^{p,q} := \wedge^p C^{2n} \otimes \wedge^q (C^{2n})^* \quad (0 \leq p, q \leq 2n).\]

Actually, it is readily seen that the operator \(T\) induces an \(\mathfrak{sp}(n, C)\)-equivalence, namely we have:

**Lemma 4.1.** For \(0 \leq l \leq 4n\), define the representation \(\rho_l = \wedge^l (\text{ad}|_{e_i})\) acting on \(\wedge^l p_C\), and recall that \(\varphi_n\) denotes the standard complex representation of \(\mathfrak{sl}_n = \mathfrak{sp}(n, C)\) on \(C^{2n}\). There is a \(\mathfrak{sl}_n\)-module equivalence:

\[\big(\rho_l, \wedge^l p_C\big) \sim \bigoplus_{p+q=l} \big(\pi_{p,q}, \wedge^{p,q}\big)\]

where we have set

\[\pi_{p,q} := \wedge^p \varphi_n \otimes \wedge^q \varphi_n^* \quad (0 \leq p, q \leq 2n).\]

We must now examine how the complementary part \(\mathfrak{sl}_0\) in \(\mathfrak{sl}_n\) acts on this decomposition. For this purpose, let us regard \(\mathfrak{sl}_0\) as \(\mathfrak{sl}(2, C)\), and let then \((X, H, Y)\) denote the ‘usual’ triple of generators of \(\mathfrak{sl}_0\). Reminding notation (2.8), we mean that \(X\) (resp. \(Y\)) is a root vector for the positive (resp. negative) root \(2\varepsilon_{n+1}\) (resp. \(-2\varepsilon_{n+1}\)) of \(\mathfrak{sl}_0\), that \(H\) is the member of the Cartan subalgebra (consisting of diagonal elements) in \(\mathfrak{sl}_0\) associated with \(2\varepsilon_{n+1}\) by duality with respect to the Killing form, and that everything is normalized in order to obtain the classical rules

\[\[H, X]\] = 2X, \quad \[H, Y]\] = -2Y, \quad \[X, Y]\] = H.

The following result is then immediate.

**Lemma 4.2.** Let \((e_i)\) (resp. \((e^i)\)) denote the canonical basis of \(C^{2n}\) (resp. \((C^{2n})^*)\), the \(e_i\)’s and \(e^i\)’s being viewed as elements of \(p_C\) by means of (4.2). We have the identities:

\[(4.4) \quad \text{ad}(X)e_i = \begin{cases} -e_i^{i+n} & \text{if } i \leq n, \\ e_i^{i-n} & \text{if } i > n, \end{cases} \quad \text{ad}(X)e^i = 0 \quad (\forall i),\]

\[(4.5) \quad \text{ad}(H)e_i = -e_i \quad (\forall i), \quad \text{ad}(H)e^i = e^i \quad (\forall i),\]

\[\text{ad}(Y)e_i = 0 \quad (\forall i), \quad \text{ad}(Y)e^i = \begin{cases} e_i^{i+n} & \text{if } i \leq n, \\ e_i^{i-n} & \text{if } i > n. \end{cases}\]

Now we need to introduce a particular \(\mathfrak{sl}_0\)-representation.

**Proposition 4.3** ([Mös83]). Consider the endomorphism \(\Phi : \wedge^l p_C \to \wedge^l p_C\) defined by \(\Phi(v) = \tau_l(X)v\) and let \(0 \leq p, q \leq 2n\) be such that \(p + q = l\). Then:

1. \(\Phi\) intertwines the \(\mathfrak{sl}_0\)-modules \(\pi_{p,q}\) and \(\pi_{p-1,q+1}\); in particular, the subspace \(\ker \Phi|_{\wedge^l p_C}\)
   defines a \(\mathfrak{sl}_0\)-representation which will be denoted by \(R_{p,q}\);
2. If \(p > q\), \(\Phi|_{\wedge^l p_C}\) is injective, hence \(R_{p,q}\) is trivial;
3. If \(p \leq q\), \(\Phi|_{\wedge^l p_C}\) is surjective, hence \(R_{p,q} = \pi_{p,q} - \pi_{p-1,q+1}\).

(In these statements, we adopt the convention \(\pi_{-1,l+1} = 0\).)
Lemma 4.5. By comparing (4.9) and (4.10), we must have conditions: there exists a basis \((\lambda)\) for finite dimensional representations of \(\pi^k\). Proof. From (4.4) it is easily seen that \(\Phi\) maps \(\wedge^{p,q}\) to \(\wedge^{p-1,q+1}\). Besides, if \(Z \in \mathfrak{P}_C\), then \([X, Z] = 0\), so that \(\Phi \in \text{End}_{\mathfrak{P}}(\wedge^{p,C})\). Hence (1) is proved.

Next, in order to prove (2), assume that \(p > q\) and that there exists a nonzero vector \(v \in \wedge^{p,q}\) such that \(\Phi(v) = 0\). We view \(\wedge^{p,C}\) as a \(\mathfrak{P}_C\)-module and we let \(V\) denote the \(\mathfrak{P}_C\)-module generated by \(v\). It decomposes as a direct sum

\[
V = \bigoplus_{i \in I} V_i
\]

of irreducible representations \(V_i\) of \(\mathfrak{P}_C\). Writing \(v = \sum v_i\) accordingly, the assumption \(\Phi(v) = 0\) gives

\[
\Phi(v_i) = 0, \quad \forall i.
\]

Before to proceed with our proof, we recall the well-known description of irreducible finite dimensional representations of \(\mathfrak{P}_C \simeq \mathfrak{sl}(2, \mathbb{C})\) (we take the formulation of [Wal73], §4.3.10).

**Lemma 4.4.** Any irreducible finite dimensional representation of \(\mathfrak{P}_C \simeq \mathfrak{sl}(2, \mathbb{C})\) is equivalent to a representation \((\pi_k, H_k)\) \((k \geq 0)\) of dimension \(k + 1\), defined by the following conditions: there exists a basis \((\varepsilon_1, \ldots, \varepsilon_k)\) of \(H_k\) such that

\[
\begin{align*}
\pi_k(H)\varepsilon_j &= (k - 2j)\varepsilon_j, \\
\pi_k(X)\varepsilon_j &= -j\varepsilon_{j-1} \text{ if } j > 0, \\
\pi_k(X)\varepsilon_0 &= 0, \\
\pi_k(Y)\varepsilon_j &= -(k - j)\varepsilon_{j+1} \text{ if } j < k, \\
\pi_k(Y)\varepsilon_k &= 0.
\end{align*}
\]

Now, since each \(V_i\) is some \(H_{k_i}\), we can write \(v_i = \sum_{j=0}^{k_i} \lambda^i_j \varepsilon^i_j\) where \((\varepsilon^i_1, \ldots, \varepsilon^i_{k_i})\) is a basis of \(H_{k_i}\) which satisfies the conditions of the lemma. Then (4.8) and (4.6) imply that \(v_i = \lambda^i_0 \varepsilon^i_0\), which in turns yields

\[
\tau(H)v_i = k_i v_i, \quad \text{with } k_i = \text{dim } H_{k_i} - 1 \geq 0
\]

because of (4.7). On the other hand, an easy calculation with the formulas of (4.5) gives us

\[
\tau(H)v = (q - p)v.
\]

By comparing (4.9) and (4.10), we must have \(k_i = q - p\) for all \(i\), hence \(q - p \geq 0\), which brings the contradiction and proves (2).

Assertion (3) follows from similar considerations, which are left to the reader. □

Reminding notation of Section 2, let \(\tau_{r,s,t}\) denote the irreducible representation of \(\mathfrak{P}_C\) whose highest weight is

\[
\lambda_{r,s,t} = \sum_{k=1}^{r} 2\varepsilon_k + \sum_{k=r+1}^{r+s} \varepsilon_k + t\varepsilon_{n+1} \quad (r, s, t \in \mathbb{N}, \ r + s \leq n).
\]

Likewise, we define \(\tau_{r,s}\) as the irreducible representation of \(\mathfrak{P}_C^0 = \mathfrak{sp}(n, \mathbb{C})\) with highest weight

\[
\lambda_{r,s} = \sum_{k=1}^{r} 2\varepsilon_k + \sum_{k=r+1}^{r+s} \varepsilon_k. \quad (r, s \in \mathbb{N}, \ r + s \leq n).
\]

**Lemma 4.5.** (1) For \(0 \leq p, q \leq 2n\), any irreducible summand of the \(\mathfrak{P}_C\)-modules \(\tau_{p,q}\) and \(R_{p,q}\) is some \(\tau_{r,s}\).
For $0 \leq l \leq 4n$, any irreducible summand of the $\mathfrak{k}_C$-module $\tau_l$ is some $\tau_{r,s,t}$, with $t \leq l$ and $t, l$ having same parity.

Proof. Let $0 \leq p, q \leq 2n$. Since $R_{p,q} = \pi_{p,q} - \pi_{p-1,q+1}$, it is enough to prove (1) for the representation $\pi_{p,q}$. The obvious equivalences

$$\wedge^p \varphi_n \sim \wedge^{2n-p} \varphi_n$$

and

$$\varphi_n \sim \varphi_n^*$$

imply the following ones:

(4.13) \hspace{1cm} \pi_{p,q} \sim \pi_{2n-p,q} \sim \pi_{p,2n-q},

(4.14) \hspace{1cm} \pi_{p,q} \sim \wedge^p \varphi_n \otimes \wedge^q \varphi_n.

In particular, we can, and shall, assume $p, q \leq n$. Then (1) follows immediately from the formula

$$\wedge^p \varphi_n = \bigoplus_{k=0}^{[p/2]} \tau_{0,p-2k}$$

(see e.g. [Bou75], Ch. VIII, §13.3).

Next, we know that any irreducible representation of $\mathfrak{k}_C$ must be some tensor product of an irreducible representation of $\mathfrak{k}_C^0$ by an irreducible representation of $\mathfrak{k}_C^1$. By (1), the irreducible summands of $\tau_l$ are thus of the form $\tau_{r,s} \otimes \sigma$, with $\sigma$ an irreducible representation of $\mathfrak{k}_C^1 = \mathfrak{sp}(1, \mathbb{C})$. But these are known by Lemma 4.4, and the proof of this result shows actually that $\sigma$ must be the symmetric power $S^t \varphi_1$ of the standard representation $\varphi_1$ for some $t \in \mathbb{N}$. With our notation, $S^t \varphi_1$ has highest weight $t \varepsilon_{n+1}$, hence we have proved that any irreducible component of $\tau_l$ must be some $\tau_{r,s,t}$. As concerns the conditions on $t$, they follow easily by observing that formula (4.1) will yield irreducible factors to which $\mathfrak{k}_C^1$ contributes by tensor products of symmetric powers of $\varphi_1$, and these are governed by the classical Clebsch-Gordan formula

$$S^j \varphi_1 \otimes S^k \varphi_1 = \bigoplus_{i=0}^{\min(j,k)} S^{i+k-2j} \varphi_1.$$

This finishes the proof of our lemma. \hfill \Box

The following correspondence is the main result of [Mös83].

**Theorem 4.6 ([Mös83]).** For $0 \leq l \leq 4n$, let

$$\Pi_l = \bigoplus_{p+q=l \leq 2n} R_{p,q}.$$

There is a one-to-one correspondence between the irreducible components of the $\mathfrak{k}_C$-module $\tau_l$ and the irreducible components of the $\mathfrak{k}_C^0$-module $\Pi_l$, namely:

1. If $\tau_{r,s}$ is an irreducible factor of $R_{p,q} \subset \Pi_{p+q}$, then $\tau_{r,s,q-p}$ is an irreducible factor of $\tau_{p+q}$;

2. If $\tau_{r,s,t}$ is an irreducible factor of $\tau_l$, then $\tau_{r,s}$ is an irreducible factor of $R_{\frac{t-l}{2}, \frac{t+l}{2}}$.

Proof. Let $\nu$ be a highest weight vector for some $\tau_{r,s} \subset R_{p,q} \subset \Pi_{p+q}$. By definition of $R_{p,q}$ in Proposition 4.3, we have $\tau_{p+q}(X)\nu = 0$, so that $\nu$ must be also a highest weight vector for some irreducible factor of $\tau_{p+q}$. From the relation (4.10) we see that this irreducible factor is $\tau_{r,s,q-p}$.

Conversely, assume that $\tau_{r,s,t}$ is an irreducible component of $\tau_l$ and let $\nu$ be a highest weight vector for $\tau_{r,s,t}$. We have the identity $\tau_l(H)\nu = t\nu$, so by decomposing $\nu$ according
to (4.3) and by using (4.10) we see that \( v \in \bigwedge^{l \leq 1, l \leq t} \). Moreover we know that \( \Phi(v) = \tau_l(X)v = 0 \), thus \( v \) must be a highest weight vector for some irreducible component of \( R^{l \leq t, l \leq t} \), and this component is clearly \( \tau_{r,s} \).

We shall actually give a more convenient and more explicit statement of this theorem. Let us recall first two well-known decompositions.

**Proposition 4.7.**  
(1) If \( 0 \leq p, q \leq n \), then

\[
\pi_{p,q} = \bigoplus_{u=0}^{[p/2]} \bigoplus_{v=0}^{[q/2]} \tau_{0,p-2u} \otimes \tau_{0,q-2v}.
\]

(2) For \( 0 \leq r, s \leq n \), the decomposition of \( \tau_{0,r} \otimes \tau_{0,s} \) into irreducible factors is given by the formula

\[
\tau_{0,r} \otimes \tau_{0,s} = \bigoplus_{j=(r+s-n)+}^{\min(r,s)} \bigoplus_{i=0}^{j} \tau_{i-j,r+s-2j},
\]

where \( (a)_+ = \max(a, 0) \), and with the convention that \( \bigoplus_{j=a}^{b} = 0 \) if \( b < a \).

**Proof.** Assertion (1) is an immediate consequence of (4.14) and (4.15), and assertion (2) is a rephrasing of [OV90], Table 5 p. 302.

Thus, the results of Proposition 4.7 allow us to decompose into irreducible summands the representations \( \pi_{p,q} \) and \( R_{p,q} = \pi_{p,q} - \pi_{p-1,q+1} \). However, we must pay attention to the fact that we cannot employ strictly speaking these decompositions for all values of \( p, q \) such that \( p \leq q \) and \( p + q = l \leq 2n \) (from now on, it will be understood that we can restrict our discussion to \( l \leq 2n \) by Hodge duality). Let us elaborate.

Assume first that \( p = 0 \). Then the decomposition of \( R_{0,q} = \pi_{0,q} \) is given by (4.16) when \( q \leq n \). But if \( q \geq n + 1 \), we must use the equivalence

\[
R_{0,q} = \pi_{0,q} \sim \pi_{0,2n-q} = R_{0,2n-q}
\]

which comes from (4.13). Next, assume that \( 1 \leq p \leq q \leq n - 1 \). Then (4.16) can be applied both to \( \pi_{p,q} \) and \( \pi_{p-1,q+1} \) and yields therefore the decomposition of \( R_{p,q} \) into irreducibles. Lastly, suppose that \( 1 \leq p \leq n \leq q \) (hence \( q \leq 2n - 1 \)). Using again (4.13), we write in this case

\[
R_{p,q} = \pi_{p,q} - \pi_{p-1,q+1} \sim \pi_{p,2n-q} - \pi_{p-1,2n-1}.
\]

Motivated by this discussion we put

\[
R'_{p,q} = \begin{cases} 
\pi_{p,q} - \pi_{p-1,q+1} & \text{if } 1 \leq p \leq q \leq n, \\
\pi_{p,q} & \text{if } p \leq q \leq n \text{ and } pq = 0.
\end{cases}
\]

so that we have

\[
R_{p,q} \sim R'_{p,2n-q} \text{ for } q \geq n.
\]

Besides, (4.16) implies that

\[
\pi_{p-1,q+1} = \pi_{p-1,q+1} - \bigoplus_{u=0}^{[p-1/2]} \tau_{0,p-1-2u} \otimes \tau_{0,q+1},
\]

for \( 1 \leq p \leq q \leq n - 1 \). Actually, this formula extends to the cases \( 0 \leq p \leq q \leq n \), since we have set \( \pi_{-1,k} = 0 \) and since it is natural to declare that \( \tau_{0,n+1} = 0 \) (remind (4.12)).
This last convention will be reflected anyway by the fact that \( \tau_{0,p-1-2u} \otimes \tau_{0,n+1} = 0 \) for all \( p \) in virtue of (4.17). Therefore, we have

\[
(4.18) \quad R'_{p,q} = R_{p,q} \bigoplus \bigoplus_{u=0}^{\lceil \frac{q-1}{2} \rceil} \tau_{0,p-1-2u} \otimes \tau_{0,q+1}, \quad \text{for all } 0 \leq p \leq q \leq n.
\]

Let us define now the two following sets, for \( 0 \leq p \leq q \leq n \):

\[
(4.19) \quad I_{p,q} = \{(r, s) \in \{0, \ldots, p\} \times \{0, \ldots, q\} \mid \tau_{r,s} \text{ is a summand of } R_{p,q}\};
\]

\[
(4.20) \quad I'_{p,q} = \{(r, s) \in \{0, \ldots, p\} \times \{0, \ldots, q\} \mid \tau_{r,s} \text{ is a summand of } R'_{p,q}\}.
\]

As concerns these notations we assume that a pair \((r, s) \in I_{p,q} \) (resp. \((r, s) \in I'_{p,q}\)) is counted as much as is the multiplicity of \( \tau_{r,s} \) in \( R_{p,q} \) (resp. in \( R'_{p,q}\)).

To sum up, we can state the following rephrasing of Theorem 4.6.

**Theorem 4.8.**

1. If \( 0 \leq l \leq n \), then

\[
\tau_{l} = \bigoplus_{p+q=l, (r,s) \in I_{p,q}} \bigoplus_{p \leq q} \tau_{r,s,q-p}.
\]

2. If \( n+1 \leq l \leq 2n \), then

\[
\tau_{l} = \bigoplus_{p+q=l, (r,s) \in I_{p,q}} \bigoplus_{p \leq q} \tau_{r,s,q-p} \bigoplus \bigoplus_{p+q=l, (r,s) \in I'_{p,2n-q}} \bigoplus_{p \leq q} \tau_{r,s,q-p}.
\]

(If \( 2n \leq l \leq 4n \), we have \( \tau_{l} \sim \tau_{4n-l} \) by Hodge duality.)

From this result, we see that our problem reduces to describe the sets \( I_{p,q} \) and \( I'_{p,q} \) defined by (4.19) and (4.20) for \( 0 \leq p \leq q \leq n \), hence to decompose the \( \mathbb{F}_{0}^{\mathbb{C}} \)-modules \( R_{p,q} \) and \( R'_{p,q} \) into irreducible summands for \( 0 \leq p \leq q \leq n \). Of course, this will be done by implementing the results of Proposition 4.7, and to prepare our calculations we first emphasize certain relations which will be often used later on.

**Lemma 4.9.**

1. If \( 1 \leq r \leq s \), then

\[
(4.21) \quad \tau_{0,r} \otimes \tau_{0,s} - \tau_{0,r-1} \otimes \tau_{0,s+1} = \bigoplus_{j=0}^{r} \tau_{j,s-r}.
\]

2. If \( r = s + 1 \), then

\[
(4.22) \quad \tau_{0,r} \otimes \tau_{0,s} - \tau_{0,r-1} \otimes \tau_{0,s+1} = 0.
\]

3. If \( r \geq s + 2 \geq 2 \), then

\[
(4.23) \quad \tau_{0,r} \otimes \tau_{0,s} - \tau_{0,r-1} \otimes \tau_{0,s+1} = -\bigoplus_{j=0}^{s+1} \tau_{j,r-s-2}.
\]

and

\[
(4.24) \quad (\tau_{0,r} \otimes \tau_{0,s} - \tau_{0,r-1} \otimes \tau_{0,s+1}) \oplus (\tau_{0,s+2} \otimes \tau_{0,r} - \tau_{0,s+1} \otimes \tau_{0,r+1}) = \tau_{s+2,r-s-2}.
\]

**Proof.** Equalities (4.21) and (4.23) follow from (4.17) and imply (4.24), while (4.22) is obvious.

Together with Theorem 4.8, the following result gives a complete answer to our problem.
Theorem 4.10. Assume that \(0 \leq p \leq q \leq n\) and set

\[
U_p = \bigoplus_{u=0}^{\left\lfloor \frac{p}{2} \right\rfloor} \bigoplus_{j=0}^{u} \tau_{p-2u,2(u-j)},
\]

\[
V_{p,q} = \bigoplus_{u=0}^{\left\lfloor \frac{q}{2} \right\rfloor} \bigoplus_{v=0}^{p-2u} \tau_{j,q-p+2(u-v)},
\]

\[
V'_{p,q} = \bigoplus_{u=0}^{\left\lfloor \frac{p-1}{2} \right\rfloor} \bigoplus_{j=0}^{p-1-2u} \bigoplus_{i=0}^{j} \tau_{j-i,p+q-2(u+j)}.
\]

Then

\[
R'_{p,q} = R_{p,q} \oplus V'_{p,q}
\]

and the decomposition of \(R_{p,q}\) into \(\mathfrak{t}_0\)-irreducible factors is as follows:

1. If \(p\) and \(q\) are even, then

\[
R_{p,q} = \bigoplus_{u=0}^{\left\lfloor \frac{q}{2} \right\rfloor} \bigoplus_{v=0}^{p-2u} \tau_{0,q-2v}.
\]

2. If \(p\) is even and \(q\) is odd, then

\[
R_{p,q} = \bigoplus_{u=0}^{\left\lfloor \frac{q-1}{2} \right\rfloor} \tau_{0,q-2v}.
\]

3. If \(p\) and \(q\) are odd, then

\[
R_{p,q} = \bigoplus_{u=0}^{\left\lfloor \frac{q-1}{2} \right\rfloor} \bigoplus_{v=0}^{p-1-2u} \tau_{1,p-1-2u}.
\]

4. If \(p\) is odd and \(q\) is even, then

\[
R_{p,q} = \tau_{p,q}.
\]

In all these results, we adopt the convention that \(\bigoplus_{k=a}^{b} = 0\) if \(b < a\).

Proof. First of all, (4.25) follows clearly from (4.18) and (4.17), so that our job consists essentially in decomposing \(R_{p,q}\).

When \(p = 0\), we have \(R_{0,q} = \tau_{0,q} = \bigoplus_{v=0}^{\left\lfloor q/2 \right\rfloor} \tau_{0,q-2v}\) by (4.16). Reminding our convention in the theorem, this identity coincides with those stated in (4.26) and (4.27).

From now on, we assume that \(1 \leq p \leq q \leq n\) and proceed with a case-by-case argument.

1. Case \(p, q\) even. By (4.16), \(R_{p,q} = \pi_{p,q} - \pi_{p-1,q+1}\) can be rewritten as

\[
R_{p,q} = \bigoplus_{u=0}^{p-1} \bigoplus_{v=0}^{q/2} (\tau_{0,p-2u} \otimes \tau_{0,q-2v} - \tau_{0,p-1-2u} \otimes \tau_{0,q-2v} + \tau_{0,q-2v}).
\]

Let \(W_{p,q}\) denote the double sum in the right hand side of the previous expression. We split it into two parts

\[
W^+_{p,q} \oplus W^-_{p,q}.
\]
where in $W_{p,q}^+$, resp. in $W_{p,q}^-$, the pair $(u, v)$ ranges through the set

$$E_{p,q}^+ = \{ u = 0, \ldots, \frac{p}{2} - 1; v = 0, \ldots, \frac{q}{2}; p - 2u \leq q - 2v \},$$

resp. $E_{p,q}^- = \{ u = 0, \ldots, \frac{p}{2} - 1; v = 0, \ldots, \frac{q}{2}; p - 2u \geq q - 2v + 2 \}$.

(The case $p - 2u = q - 2v + 1$ cannot occur since $p$ and $q$ have the same parity.) Thus $W_{p,q}^-$ contains factors of the form $\tau_{0,r} \otimes \tau_{0,s} - \tau_{0,r-1} \otimes \tau_{0,s+1}$ with $r \geq s + 2$, which contribute negatively by (4.23). But, according to (4.24) these negative factors will be killed by positive factors coming from $W_{p,q}^+$, the result being some representation $\tau_{s+2,r-s-2}$. More precisely, we have, for $(u, v) \in E_{p,q}^-$:

$$\tau_{0,p-2u} \otimes \tau_{0,q-2v} - \tau_{0,p-1-2u} \otimes \tau_{0,q+1-2v} \oplus (\tau_{0,q+2-2v} \otimes \tau_{0,p-2u} - \tau_{0,q+1-2v} \otimes \tau_{0,p+1-2u}) = \tau_{q+2-2v,p-q+2(v-u-1)}.$$  

(4.31)

We insist on the fact that this identity can be applied to all $(u, v) \in E_{p,q}^-$, since

$$(u, v) \in E_{p,q}^- \Rightarrow (v + \frac{p-q}{2} - 1, u + \frac{q-p}{2}) \in E_{p,q}^+,$$

so that the terms $\tau_{0,q+2-2v} \otimes \tau_{0,p-2u} - \tau_{0,q+1-2v} \otimes \tau_{0,p+1-2u}$ in (4.31) really occur as members of $W_{p,q}^+$ in (4.30). This discussion shows that we must determine the factors in $W_{p,q}^+$ which are not affected by factors in $W_{p,q}^-$ and, for those which are affected accordingly to (4.31), what remains at the end.

Keeping (4.31) in mind, let us define new parameters $u', v'$ by

$$\begin{cases} p - 2u' = q + 2 - 2v \\ q - 2v' = p - 2u \end{cases} \quad \Rightarrow \quad \begin{cases} u' = v + \frac{p-q}{2} - 1 \\ v' = u + \frac{q-p}{2} \end{cases}$$

In the canonical plane $\mathbb{R}^2$, this change of variables corresponds to a transformation $f$ which is the symmetry about the line $v = u + \frac{q-p}{2}$ followed by the translation of vector $(-1, 0)$. It is easy to observe that

$$E_{p,q}^+ \setminus f(E_{p,q}^-) = \{ u = 0, \ldots, \frac{p}{2} - 1; v = 0, \ldots, \frac{q-p}{2} - 1 \}.$$  

As a consequence, we get

$$W_{p,q} = \bigoplus_{u=0}^{\frac{p}{2}-1} \bigoplus_{v=0}^{\frac{q}{2}-1} (\tau_{0,p-2u} \otimes \tau_{0,q-2v} - \tau_{0,p-1-2u} \otimes \tau_{0,q+1-2v}) \oplus \bigoplus_{(u,v) \in E_{p,q}^-} \tau_{q+2-2v,p-q+2(v-u-1)}.$$

By (4.21), the double sum above equals $V_{p,q}$. On the other hand,

$$\bigoplus_{(u,v) \in E_{p,q}^-} \tau_{q-2v+2,p-q+2(v-u-1)} = \bigoplus_{v=\frac{q+p}{2}+1}^{\frac{q}{2}-1} \bigoplus_{u=0}^{\frac{q-p}{2}+1} \tau_{q+2-2v,p-q+2(v-u-1)}$$

$$= \bigoplus_{v=0}^{\frac{q}{2}-1} \bigoplus_{j=0}^{\frac{p}{2}-1} \tau_{p-2v,2(v-j)}.$$

Remembering (4.30), we have finally proved (4.26).
(2) Case $p$ even, $q$ odd. By (4.16), we have
\begin{equation}
R_{p,q} = \bigoplus_{u=0}^{\frac{q}{2}-1} \bigoplus_{v=0}^{\frac{q}{2}-1} (\tau_{0,p-2u} \otimes \tau_{0,q-2v} - \tau_{0,p-1-2u} \otimes \tau_{0,q+1-2v})
\end{equation}
(4.32)
\begin{equation}
\bigoplus_{v=0}^{\frac{q}{2}-1} \tau_{0,q-2v} - \bigoplus_{u=0}^{\frac{q}{2}-1} \tau_{0,p-1-2u}.
\end{equation}

Let $W_{p,q}$ denote the double sum in the right hand side of the previous expression. This time, we split it into three parts
\begin{equation}
W_{p,q}^+ \oplus W_{p,q}^0 \oplus W_{p,q}^-,
\end{equation}
where in $W_{p,q}^+$, resp. $W_{p,q}^0$, resp. $W_{p,q}^-$, the pair $(u, v)$ ranges through the set
\begin{equation}
E_{p,q}^+ = \{ u = 0, \ldots, \frac{p}{2} - 1; v = 0, \ldots, \frac{q-1}{2}; p - 2u \leq q - 2v - 1 \},
\end{equation}
resp. $E_{p,q}^0 = \{ u = 0, \ldots, \frac{p}{2} - 1; v = 0, \ldots, \frac{q-1}{2}; p - 2u = q - 2v + 1 \}$, 
resp. $E_{p,q}^- = \{ u = 0, \ldots, \frac{p}{2} - 1; v = 0, \ldots, \frac{q-1}{2}; p - 2u \geq q - 2v + 3 \}$.
By (4.22), we have
\begin{equation}
W_{p,q}^0 = 0.
\end{equation}

We now take care of the two remaining parts as in case (1). Here, $W_{p,q}^-$ contains factors of the type $\tau_{r,s} \otimes \tau_{s,t} - \tau_{r,t-1} \otimes \tau_{s,t+1}$ with $r \geq s + 3$, which contribute negatively by (4.23). All these negative factors will be canceled by positive factors coming from $W_{p,q}^+$, as shown by the following obvious identity:
\begin{equation}
(\tau_{0,p-2u} \otimes \tau_{0,q-2v} - \tau_{0,p-1-2u} \otimes \tau_{0,q+1-2v}) \oplus (\tau_{0,q+1-2v} \otimes \tau_{0,p-1-2u} - \tau_{0,q-2v} \otimes \tau_{0,p-2u}) = 0.
\end{equation}

We change variables by setting
\begin{equation}
\begin{cases}
p - 2u' = q + 1 - 2v \\
q - 2v' = p - 1 - 2u
\end{cases}
\end{equation}
i.e.
\begin{equation}
\begin{cases}
u = v + \frac{p-q-1}{2} \\
v' = u + \frac{p-q+1}{2}
\end{cases}
\end{equation}
In the canonical plane $\mathbb{R}^2$, this corresponds to the symmetry $g$ about the line $v = u + \frac{p-q+1}{2}$. Since we have
\begin{equation}
E_{p,q}^+ \setminus g(E_{p,q}^-) = \{ u = 0, \ldots, \frac{p}{2} - 1; v = 0, \ldots, \frac{q-1}{2} \},
\end{equation}
we find that $W_{p,q} = V_{p,q}$. To finish the proof of (4.27), it remains to calculate the other term in (4.32). But it is clear that
\begin{equation}
\bigoplus_{v=0}^{\frac{q}{2}-1} \tau_{0,q-2v} - \bigoplus_{u=0}^{\frac{q}{2}-1} \tau_{0,p-1-2u} = \bigoplus_{v=0}^{\frac{q}{2}-1} \tau_{0,q-2v}.
\end{equation}

(3) Case $p, q$ odd. By (4.16), we have
\begin{equation}
R_{p,q} = \bigoplus_{u=0}^{\frac{p-1}{2}} \bigoplus_{v=0}^{\frac{p-1}{2}} (\tau_{0,p-2u} \otimes \tau_{0,q-2v} - \tau_{0,p-1-2u} \otimes \tau_{0,q+1-2v}) - \bigoplus_{u=0}^{\frac{p-1}{2}} \tau_{0,p-1-2u}.
\end{equation}
(4.33)

This case is very similar to the first one, so that we employ the same notation and the same method. The main difference is that
\begin{equation}
E_{p,q}^+ \setminus f(E_{p,q}^-) = \{ u = 0, \ldots, \frac{p-1}{2}; v = 0, \ldots, \frac{q-p-1}{2} \} \cup \{ u = \frac{p-1}{2}; v = \frac{q-p}{2}, \ldots, \frac{q-1}{2} \}.
\end{equation}
Let us denote by $J_{p,q}^+$ the second set in the right hand side of this expression. Then we get as in case (1)

$$W_{p,q} = V_{p,q} \oplus \bigoplus_{u=0}^{p-3} u \tau_{p-2u,2(u-j)} \oplus \bigoplus_{(u,v) \in J_{p,q}^+} (\tau_{0,p-2u} \otimes \tau_{0,q-2v} - \tau_{0,p-1-2u} \otimes \tau_{0,q+1-2v}).$$

But, with (4.21) we obtain

$$\bigoplus_{(u,v) \in J_{p,q}^+} (\tau_{0,p-2u} \otimes \tau_{0,q-2v} - \tau_{0,p-1-2u} \otimes \tau_{0,q+1-2v})$$

$$= \bigoplus_{v=\frac{q-p}{2}}^{q-1} (\tau_{0,1} \otimes \tau_{0,q-2v} - \tau_{0,0} \otimes \tau_{0,q+1-2v})$$

$$= \bigoplus_{v=\frac{q-p}{2}}^{q-1} (\tau_{1,q-1-2v} \oplus \tau_{0,q-1-2v})$$

$$= \bigoplus_{u=0}^{p-1} (\tau_{1,p-1-2u} \oplus \tau_{0,p-1-2u}).$$

Reminding (4.33), we thus have

$$\bigoplus_{(u,v) \in J_{p,q}^+} (\tau_{0,p-2u} \otimes \tau_{0,q-2v} - \tau_{0,p-1-2u} \otimes \tau_{0,q+1-2v}) - \bigoplus_{u=0}^{p-1} \tau_{0,p-1-2u} = \bigoplus_{u=0}^{p-1} \tau_{1,p-1-2u},$$

and this finally proves (4.28).

\textit{(4) Case $p$ odd, $q$ even.} Here we have

$$R_{p,q} = \bigoplus_{u=0}^{p-1} \bigoplus_{v=0}^{\frac{q}{2}} (\tau_{0,p-2u} \otimes \tau_{0,q-2v} - \tau_{0,p-1-2u} \otimes \tau_{0,q+1-2v}).$$

Proceeding exactly as in case (2), we get easily (4.29). □

Let us make our formulas become explicit in a few examples, by treating the cases $1 \leq l \leq 6$. 
Proposition 4.11. Assume (as usual) that \( n \geq 2 \). For \( l = 1, \ldots, 6 \), the decomposition of \( \tau_l \) into \( K \)-types is as follows:

\[
\begin{align*}
\tau_1 &= \tau_{0,1,1}; \\
\tau_2 &= \tau_{0,2,2} \oplus \tau_{0,0,2} \oplus \tau_{1,0,0}; \\
\tau_3 &= \tau_{0,3,3} \oplus \tau_{0,1,3} \oplus \tau_{0,1,1} \oplus \tau_{1,1,1}; \\
& \quad \text{if } n \geq 3 \\
\tau_4 &= \tau_{0,4,4} \oplus \tau_{0,2,4} \oplus \tau_{0,0,4} \oplus \tau_{0,2,2} \oplus \tau_{1,2,2} \oplus \tau_{1,0,2} \oplus \tau_{2,0,0} \oplus \tau_{0,2,0} \oplus \tau_{0,0,0}; \\
& \quad \text{if } n \geq 3 \\
\tau_5 &= \tau_{0,5,5} \oplus \tau_{0,3,5} \oplus \tau_{0,1,5} \oplus \tau_{0,3,3} \oplus \tau_{1,3,3} \oplus \tau_{0,1,3} \\
& \quad \text{if } n \geq 4 \quad \text{if } n \geq 3 \quad \text{if } n \geq 3 \\
& \quad \oplus \tau_{1,1,3} \oplus \tau_{0,3,1} \oplus \tau_{2,1,1} \oplus \tau_{1,1,1} \oplus \tau_{0,1,1}; \\
& \quad \text{if } n \geq 3 \\
\tau_6 &= \tau_{0,6,6} \oplus \tau_{0,4,6} \oplus \tau_{0,2,6} \oplus \tau_{0,0,6} \oplus \tau_{1,4,4} \oplus \tau_{0,4,4} \oplus \tau_{1,2,4} \oplus \tau_{0,2,4} \oplus \tau_{1,0,4} \\
& \quad \text{if } n \geq 6 \quad \text{if } n \geq 5 \quad \text{if } n \geq 4 \quad \text{if } n \geq 5 \quad \text{if } n \geq 4 \quad \text{if } n \geq 3 \\
& \quad \oplus \tau_{0,4,2} \oplus \tau_{2,2,2} \oplus \tau_{1,2,2} \oplus 2 \oplus \tau_{0,2,2} \oplus \tau_{2,0,2} \oplus \tau_{0,0,2} \oplus \tau_{1,2,0} \oplus \tau_{3,0,0} \oplus \tau_{1,0,0} \\
& \quad \text{if } n \geq 3 \quad \text{if } n \geq 3 \quad \text{if } n \geq 3 \quad \text{if } n \geq 3 \\
& \quad \text{if } n \geq 3 \quad \text{if } n \geq 3 \\
\end{align*}
\]

It is interesting to observe that multiplicities only appear for \( l \geq 6 \) (and \( n \geq 3 \)).

Proof. Since all computations are similar, we merely treat the case of \( \tau_5 \). Assume first that \( n \geq 5 \). By Theorem 4.8, we must calculate the decompositions

\[
\begin{align*}
R_{0,5} &= \tau_{0,5} \oplus \tau_{0,3} \oplus \tau_{0,1}, \\
R_{1,4} &= \tau_{0,3} \oplus \tau_{1,3} \oplus \tau_{0,1} \oplus \tau_{1,1}, \\
R_{2,3} &= \tau_{0,3} \oplus \tau_{2,1} \oplus \tau_{1,1} \oplus \tau_{0,1},
\end{align*}
\]

which follow easily from Theorem 4.10. Likewise, Theorem 4.8 tells us to write

\[
\begin{align*}
R'_{0,3} &= \tau_{0,3} \oplus \tau_{0,1}, \\
R'_{1,4} &= \tau_{0,3} \oplus \tau_{1,3} \oplus \tau_{0,1} \oplus \tau_{1,1}, \\
R'_{2,3} &= \tau_{0,3} \oplus \tau_{2,1} \oplus \tau_{1,1} \oplus \tau_{0,1},
\end{align*}
\]

when \( n = 4 \), as well as

\[
\begin{align*}
R'_{0,1} &= \tau_{0,1}, \\
R'_{1,2} &= \tau_{0,3} \oplus \tau_{1,1} \oplus \tau_{0,1}, \\
R'_{2,3} &= \tau_{0,3} \oplus \tau_{2,1} \oplus \tau_{1,1} \oplus \tau_{0,1},
\end{align*}
\]

when \( n = 3 \). Lastly, we use the fact that \( \tau_5 \sim \tau_3 \) when \( n = 2 \). \( \square \)

The results in this proposition for \( 1 \leq l \leq 5 \) were already known to A. Swann ([Swa89], also stated in [Sal89], Proposition 9.2) and E. Bonan ([Bon95a], [Bon95b]). A reader familiar with this literature may be interested in a comparison of the notations: in [Sal89], the \( K \)-representation denoted by \( \lambda_i^* \sigma^j \) equals our \( \tau_{s,r-2s,t} \), while in E. Bonan’s papers one has

\[
\begin{align*}
Q^{i,j} &= \tau_{j,i-j,i-j}, \\
Q^{p,q}_{i,j} &= \tau_{j,i-j,p-q}.
\end{align*}
\]
But E. Bonan has also given a geometrical interpretation of the representations $Q^{i,j}$: the sum

$$Q^l = \bigoplus_{i=\lfloor \frac{l}{2} \rfloor}^{\min(l,n)} Q^{l-i}$$

is none other than the subspace of hypereffective $l$-forms (as defined in Section 2). In other words:

**Proposition 4.12.** For $0 \leq l \leq 2n$, the direct sum

$$\bigoplus_{k=(l-n)_{+}}^{\lfloor \frac{l}{2} \rfloor} L^2(G, \tau_{k,l-2k,l-2k})$$

constitutes exactly the subspace of $L^2$ hypereffective $l$-forms.

Let us end this section with the proof of Theorem 1.5. Using notation of Theorem 3.3, we must identify the $K$-representations $\tau_{k}$ whose highest weights are the $\lambda_{k} = w_k \cdot 2\delta_g - 2\delta_k$ for $1 \leq k \leq m$. Here, $m = n + 1$ and for convenience we rather let the parameter $k$ range through the set $\{0, \ldots, n\}$. Computing explicitly the weights $\lambda_k$ in terms of the $\varepsilon_i$'s defined in Section 2 is easy (see e.g. Lemma 11.17 of [Bal80]) and with notation of (4.11) we obtain

$$\lambda_k = \sum_{i=1}^{k} 2\varepsilon_i + 2(n-k)\varepsilon_{n+1} = \lambda_{k,0,2(n-k)} \quad (0 \leq k \leq n).$$

Together with Proposition 4.12, this discussion implies a more precise version of Theorem 1.5:

**Theorem 4.13.** The space of $L^2$ harmonic $l$-forms on $H^n(\mathbb{H})$ is trivial, unless $l = 2n$, in which case it consists of the sum

$$\bigoplus_{k=0}^{n} L^2(G, \tau_{k,0,2n-2k})_{\Delta},$$

where $L^2(G, K, \tau_{k,0,2n-2k})_{\Delta}$ denotes the harmonic part of $L^2(G, K, \tau_{k,0,2n-2k})$ and coincides with the Hilbert space $H_k$ of the discrete series representation $\pi_k$ defined in Section 3. Moreover, the last summand

$$L^2(G, \tau_{n,0,0})_{\Delta} = H_n$$

is exactly the harmonic part of the subspace of $L^2$ hypereffective $2n$-forms on $H^n(\mathbb{H})$.

5. **The $M$-decomposition of the representation $\tau_{r,s,t}$**

We keep notation of (2.12) and (2.13) to parametrize highest weights of irreducible representations of $K$ and $M$. For convenience of the reader, let us recall the following branching rule.

**Theorem 5.1 ([Bal79]).** Let $\lambda = \sum_{j=1}^{n+1} a_j \varepsilon_j$, with $a_1 \geq \cdots \geq a_n$ and $a_j \in \mathbb{N}$ for all $j$, be the highest weight of $\tau_{\lambda} \in \hat{K}$. Let $\mu = b_0 (\varepsilon_1 + \varepsilon_{n+1}) + \sum_{j=2}^{n} b_j \varepsilon_j$, with $2b_0 \in \mathbb{N}$,
b_2 \geq \cdots \geq b_n and b_j \in \mathbb{N} for all j \geq 2, be the highest weight of \( \sigma_\mu \in \hat{M} \). Define

\[
A_1 = a_1 - \max(a_2, b_2), \\
A_2 = \min(a_2, b_2) - \max(a_3, b_3), \\
\vdots \\
A_{n-1} = \min(a_{n-1}, b_{n-1}) - \max(a_n, b_n), \\
A_n = \min(a_n, b_n).
\]

Then \( \tau_\lambda|_M \supset \sigma_\mu \) if and only if

1. \( a_j \geq b_{j+1} \) for \( 1 \leq j \leq n-1 \),
2. \( b_j \geq a_{j+1} \) for \( 2 \leq j \leq n-1 \), and
3. \( b_0 = \frac{a_{n+1} + b_1 - 2l}{2} \) for some \( l \in \{0, \ldots, \min(a_{n+1}, b_1)\} \), where \( b_1 \) satisfies \( b_1 \in \mathbb{N} \) and \( \sum_{j=1}^n (a_j + b_j) \in 2\mathbb{N} \).

If these conditions hold then the multiplicity of \( \sigma_\mu \) in \( \tau_\lambda|_M \) equals

\[
m(\mu, \lambda) := \sum_{b_1 \text{ satisfying (3)}} \tilde{m}(\mu, \lambda),
\]

where

\[
\tilde{m}(\mu, \lambda) = \sum_{L \subseteq \{1, \ldots, n\}} (-1)^{|L|} \left( n - 2 - |L| + \frac{1}{2}(-b_1 + \sum_{j=1}^n A_j) - \sum_{j \in L} A_j \right).
\]

(|L| is the cardinality of L and by convention \( \left( \frac{x}{y} \right) = 0 \) if \( x - y \notin \mathbb{N} \)).

Now we use this result to describe the set \( \hat{M}(\tau_{r,s,t}) \), i.e. we give the \( M \)-decomposition of the \( K \)-irreducible representation \( \tau_{r,s,t} \) defined by the highest weight \( \lambda_{r,s,t} \) in (4.11).

**Theorem 5.2.** For \( a, b \in \mathbb{N} \) with \( a + b \leq n \) and \( 2c \in \mathbb{N} \), let \( \sigma_{a,b,c} \) denote the irreducible \( M \)-representation corresponding to the highest weight

\[
\mu_{a,b,c} = \begin{cases} 
    c(\epsilon_1 + \epsilon_{n+1}) + \sum_{i=2}^a 2\epsilon_i + \sum_{i=a+1}^{a+b} \epsilon_i & \text{if } a > 0, \\
    c(\epsilon_1 + \epsilon_{n+1}) + \sum_{i=2}^b \epsilon_i & \text{if } a = 0.
\end{cases}
\]

Fix \( (r, s, t) \in \mathbb{N}^3 \), with \( r + s \leq n \). Then \( \hat{M}(\tau_{r,s,t}) = \{ \sigma_{1, r,s,t}^{1,1}, \ldots, \sigma_{15, r,s,t}^{1,1} \} \), where the representations \( \sigma_{1, r,s,t}^{1,1}, \ldots, \sigma_{15, r,s,t}^{1,1} \) are defined as follows (we write \( \sigma_i = \sigma_{i, r,s,t}^{1,1} \) for short):

1. \( \sigma_1 = \sigma_{r+1, s-1, \frac{1}{2}} \), occurring with multiplicity 1;
2. \( \sigma_2 = \sigma_{r+1, s-1, \frac{1}{2}} \), occurring with multiplicity 1 (resp. 0) if \( t \geq 1 \) (resp. if \( t = 0 \));
3. \( \sigma_3 = \sigma_{r+1, s-1, \frac{1}{2}} \), occurring with multiplicity 1 (resp. 0) if \( t \geq 1 \) (resp. if \( t = 0 \));
4. \( \sigma_4 = \sigma_{r+1, s-2, \frac{1}{2}} \), occurring with multiplicity 1;
5. \( \sigma_5 = \sigma_{r+1, s+1, \frac{1}{2}} \), occurring with multiplicity 1 (resp. 0) if \( t \geq 1 \) (resp. if \( t = 0 \));
6. \( \sigma_6 = \sigma_{r+1, s+1, \frac{3}{2}} \), occurring with multiplicity 1 (resp. 0) if \( t \geq 2 \) (resp. if \( t = 0, 1 \));
7. \( \sigma_7 = \sigma_{r, s+1, \frac{3}{2}} \), occurring with multiplicity 2 (resp. 1) if \( t \geq 1 \) (resp. if \( t = 0 \));
8. \( \sigma_8 = \sigma_{r, s+1, \frac{3}{2}} \), occurring with multiplicity 1 (resp. 0) if \( t \geq 1 \) (resp. if \( t = 0 \));
9. \( \sigma_9 = \sigma_{r, s+1, \frac{3}{2}} \), occurring with multiplicity 1 (resp. 0) if \( t \geq 1 \) (resp. if \( t = 0 \));
10. \( \sigma_{10} = \sigma_{r+1, s-1, \frac{1}{2}} \), occurring with multiplicity 1 (resp. 0) if \( t \geq 1 \) (resp. if \( t = 0 \));
11. \( \sigma_{11} = \sigma_{r-1, s+1, \frac{1}{2}} \), occurring with multiplicity 1;
12. \( \sigma_{12} = \sigma_{r-1, s+3, \frac{1}{2}} \), occurring with multiplicity 1;
This table must be read with the following two conventions:

- any sum of the form $\sum_{i=k}^{k-1} \varepsilon_i$ appearing in a highest weight $\mu_{a,b,c}$ must be replaced by zero;
- if a highest weight $\mu_{a,b,c}$ contains a sum of the form $\sum_{i=k}^{k-2} \varepsilon_i$ or $\sum_{i=k}^{k-3} \varepsilon_i$, then it has multiplicity 0 and can be canceled from the list above.

Before proving the theorem, let us make our conventions become more explicit by treating an example. Take for instance a representation of the form $\tau_{r-1,s,t}$, occurring with multiplicity 1 (resp. 0) if $t \geq 1$ (resp. if $t = 0$);

(14) $\sigma_{14} = \sigma_{r-1,s+1,t+1}$, occurring with multiplicity 1;

(15) $\sigma_{15} = \sigma_{r-1,s,t+1}$, occurring with multiplicity 1.

Proof of Theorem 5.2. For the simplicity of the demonstration, we shall assume that $r,s \geq 3$. For $k = 5, \ldots, 15$, $\sigma_k$ has highest weight $\lambda_{a,b,c}$ with $a = 0$, which contains a sum $\sum_{i=2}^{s} 2\varepsilon_i$ (for $k = 5, \ldots, 11$) or a sum $\sum_{j=2}^{s-1} 2\varepsilon_j$ (for $k = 12, \ldots, 15$). Thus the only irreducible summands of $\tau_{0,s,t}$ occuring with positive multiplicity are $\sigma_1, \sigma_2$ (if $t \geq 1$), $\sigma_3$ and $\sigma_4$. Moreover, the first convention says that the corresponding highest weights are respectively

$$
\begin{align*}
\mu_1 &= \frac{t}{2}(\varepsilon_1 + \varepsilon_{n+1}) + \sum_{i=2}^{s+1} \varepsilon_i, \\
\mu_2 &= \frac{t-1}{2}(\varepsilon_1 + \varepsilon_{n+1}) + \sum_{i=2}^{s} \varepsilon_i, \\
\mu_3 &= \frac{t+1}{2}(\varepsilon_1 + \varepsilon_{n+1}) + \sum_{i=2}^{s} \varepsilon_i, \\
\mu_4 &= \frac{t}{2}(\varepsilon_1 + \varepsilon_{n+1}) + \sum_{i=2}^{s-1} \varepsilon_i.
\end{align*}
$$

(Further simplifications occur if $s = 0, 1, 2$.)

Proof of Theorem 5.2. For the simplicity of the demonstration, we shall assume that $r,s \geq 3$ and $r+s < n$. This is the most general case, for which the simplifications imposed by the conventions at the end of our statement do not occur. In the remaining cases, the proof is similar (and easier).

Recall that the highest weight of $\tau_{r,s,t}$ is given by (4.11). Adopting the notation of Theorem 5.1, let $\mu = b_0(\varepsilon_1 + \varepsilon_{n+1}) + \sum_{j=2}^{s} b_j \varepsilon_j$, with $2b_0 \in \mathbb{N}$, $b_2 \geq \cdots \geq b_n$ and $b_j \in \mathbb{N}$ for all $j \geq 2$, be the highest weight of some $\sigma_\mu \in \hat{M}$ which is supposed to occur in the $M$-decomposition of $\tau_{r,s,t}$. We thus have

$$
a_1 = \cdots = a_r = 2, \quad a_{r+1} = \cdots = a_{r+s} = 1, \quad a_{r+s+1} = \cdots = a_n = 0, \quad a_{n+1} = t,
$$

which gives

(5.2)

$$
\begin{align*}
A_1 &= 2 - \max(2,b_2), \\
A_2 &= \min(2,b_2) - \max(2,b_3), \quad A_{r+s-1} = \min(1,b_{r+s-1}) - \max(1,b_{r+s}), \\
&\vdots \quad A_{r+s} = \min(1,b_{r+s}) - b_{r+s+1}, \\
A_{r-1} &= \min(2,b_{r-1}) - \max(2,b_r), \quad A_{r+s+1} = -b_{r+s+2}, \\
A_r &= \min(2,b_r) - \max(1,b_{r+1}), \quad A_{r+s+2} = -b_{r+s+3}, \\
A_{r+1} &= \min(1,b_{r+1}) - \max(1,b_{r+2}), \quad A_{n-1} = -b_n, \\
&\vdots \quad A_n = 0.
\end{align*}
$$
On the other hand, conditions (1) and (2) of Theorem 5.1 yield

\[
\begin{align*}
&\begin{cases}
  b_2 = \cdots = b_{r-1} = 2, \\
  2 \geq b_r \geq b_{r+1} \geq 1, \\
  b_{r+2} = \cdots = b_{r+s-1} = 1, \\
  1 \geq b_{r+s} \geq b_{r+s+1} \geq 0, \\
  b_{r+s+2} = \cdots = b_n = 0.
\end{cases}
\end{align*}
\]

We obtain this way 9 choices for the 4-tuple \((b_r, b_{r+1}, b_{r+s}, b_{r+s+1})\), and thus for the \((n-1)\)-tuple \((b_2, \ldots, b_n)\) which (partially) labels \(\mu\). In order to determine the possible values for the remaining parameter \(b_0\) of \(\mu\), we investigate now these cases separately.

(1) Case \((b_r, b_{r+1}, b_{r+s}, b_{r+s+1}) = (2, 2, 1, 1)\). By (5.2) we have \(A_i = 0\) for all \(i\). On the other hand, the integer \(b_1\) defined by condition (3) in Theorem 5.1 must be even: \(b_1 = 2k\), with \(k \in \mathbb{N}\). We thus get

\[
\tilde{m}(\mu, \lambda) = \sum_{L \subset \{1, \ldots, n\}} (-1)^{|L|} \binom{n - 2 - |L| - k}{n - 2}.
\]

Consequently, \(\tilde{m}(\mu, \lambda) > 0\) forces \(|L| = k = 0\), hence \(b_1 = 0\), \(b_0 = \frac{1}{2}\) and \(m(\mu, \lambda) = 1\). The corresponding representation is \(\sigma_1\).

(2) Case \((b_r, b_{r+1}, b_{r+s}, b_{r+s+1}) = (2, 2, 1, 0)\). By (5.2) we have \(A_i = 0\) for all \(i \neq r + s\) and \(A_{r+s} = 1\). Moreover we have \(b_1 = 2k + 1\) for some \(k \in \mathbb{N}\), so that

\[
\tilde{m}(\mu, \lambda) = \sum_{L \subset \{1, \ldots, n\}} (-1)^{|L|} \binom{n - 2 - |L| - k - \sum_{j \in L} A_j}{n - 2}.
\]

Hence \(\tilde{m}(\mu, \lambda) > 0\) implies that \(|L| = k = \sum_{j \in L} A_j = 0\), which yields \(b_1 = 1\) and the two possibilities \(b_0 = \frac{t+1}{2}\) (for all \(t \geq 0\)) and \(b_0 = \frac{t-1}{2}\) (if \(t \geq 1\)). These results correspond to the representations \(\sigma_2\) (for \(t \geq 1\)) and \(\sigma_3\), both occuring with multiplicity \(m(\mu, \lambda) = 1\).

(3) Case \((b_r, b_{r+1}, b_{r+s}, b_{r+s+1}) = (2, 2, 0, 0)\). Proceeding as in case 1), we obtain \(\sigma_4\).

(4) Case \((b_r, b_{r+1}, b_{r+s}, b_{r+s+1}) = (2, 1, 1, 1)\). Proceeding as in case 2), we obtain \(\sigma_5\) and \(\sigma_6\).

(5) Case \((b_r, b_{r+1}, b_{r+s}, b_{r+s+1}) = (2, 1, 1, 0)\). By (5.2), we have \(A_i = 0\) for all \(i \neq r, r + s\) and \(A_r = A_{r+s} = 1\). Moreover we have \(b_1 = 2k\) for some \(k \in \mathbb{N}\), so that

\[
\tilde{m}(\mu, \lambda) = \sum_{L \subset \{1, \ldots, n\}} (-1)^{|L|} \binom{n - 2 - |L| - k + 1 - \sum_{j \in L} A_j}{n - 2}.
\]
Hence \( \tilde{m}(\mu, \lambda) > 0 \) implies that \( |L| + k + \sum_{j \in L} A_j = 0 \) or 1. To treat this new situation, we split the expression

\[
\tilde{m}(\mu, \lambda) = \sum_{L \subseteq \{1, \ldots, n\}, r \in L, r + s \in L} (-1)^{|L|} \left( \frac{n - 2 - |L| - k - 1}{n - 2} \right) = 0
\]

\[
+ \sum_{L \subseteq \{1, \ldots, n\}, r \in L, r + s \notin L} (-1)^{|L|} \left( \frac{n - 2 - |L| - k}{n - 2} \right) = 0 \text{ since } |L| \geq 1
\]

\[
+ \sum_{L \subseteq \{1, \ldots, n\}, r \notin L, r + s \in L} (-1)^{|L|} \left( \frac{n - 2 - |L| - k + 1}{n - 2} \right) = 0 \text{ since } |L| \geq 1
\]

Therefore, \( \tilde{m}(\mu, \lambda) > 0 \) forces \( r \notin L, r + s \notin L \) and \( |L| + k = 0 \) or 1.

If \( k = 0 \), then \( L \) is empty or contains a single element which is neither \( r \) nor \( r + s \) (there are \( n - 2 \) such possibilities for \( L \)). In that case,

\[
\tilde{m}(\mu, \lambda) = \left( \frac{n - 1}{n - 2} \right) + (n - 2)(-1) \left( \frac{n - 2}{n - 2} \right) = 1.
\]

Since \( b_1 = 0 \), we get \( b_0 = \frac{t}{2} \).

Now, if \( k = 1 \), then \( L \) must be empty and \( \tilde{m}(\mu, \lambda) = 1 \). Since \( b_1 = 2 \), \( b_0 \) can take the values \( \frac{t}{2} + 1 \) (for all \( t \geq 0 \)), \( \frac{t}{2} \) (if \( t \geq 1 \)) and \( \frac{t}{2} - 1 \) (if \( t \geq 2 \)).

To sum up the two cases, we obtain

\[
b_0 = \begin{cases} 
\frac{t}{2} - 1 & \text{if } t \geq 2, \text{ and the corresponding } \mu \text{ has multiplicity } 1, \\
\frac{t}{2} & \text{if } t \geq 1, \text{ and the corresponding } \mu \text{ has multiplicity } 2, \\
\frac{t}{2} & \text{if } t = 0, \text{ and the corresponding } \mu \text{ has multiplicity } 1, \\
\frac{t}{2} + 1 & \text{if } t \geq 0, \text{ and the corresponding } \mu \text{ has multiplicity } 1.
\end{cases}
\]

The corresponding \( M \)-representations are \( \sigma_7, \sigma_8, \sigma_9 \).

(6) Case \( (b_r, b_{r+1}, b_{r+s}, b_{r+s+1}) = (2, 1, 0, 0) \). Proceeding as in case 2), we get \( \sigma_{10} \) and \( \sigma_{11} \).

(7) Case \( (b_r, b_{r+1}, b_{r+s}, b_{r+s+1}) = (1, 1, 1, 1) \). Proceeding as in case 1), we get \( \sigma_{12} \).

(8) Case \( (b_r, b_{r+1}, b_{r+s}, b_{r+s+1}) = (1, 1, 1, 0) \). Proceeding as in case 2), we get \( \sigma_{13} \) and \( \sigma_{14} \).

(9) Case \( (b_r, b_{r+1}, b_{r+s}, b_{r+s+1}) = (1, 1, 0, 0) \). Proceeding as in case 1), we get finally \( \sigma_{15} \).

Remind from Theorem 3.2 that we must determine which are, among the \( M \)-irreducible components of \( \tau_1 \), the representations \( \sigma \) whose Casimir value \( c(\sigma) \) (defined by (3.7)) is maximal. Since \( K \)-irreducible components of \( \tau_1 \) are representations of the form \( \tau_{r,s,t} \) (by the results of Section 4), our first task is naturally to compare the Casimir values of the members of \( \hat{M}(\tau_{r,s,t}) \) for a fixed \( (r, s, t) \).
Proposition 5.3. (We keep notation and convention of Theorem 5.2.)

1. If \( r = 0 \), then \( \max_{i=1,\ldots,15} c(\sigma_i) \in \{c(\sigma_1), c(\sigma_3)\} \).
2. If \( s = 0 \), then \( \max_{i=1,\ldots,15} c(\sigma_i) \in \{c(\sigma_1), c(\sigma_6), c(\sigma_9)\} \).
3. If \( r \neq 0 \) and \( s \neq 0 \), then \( \max_{i=1,\ldots,15} c(\sigma_i) \in \{c(\sigma_1), c(\sigma_3), c(\sigma_9)\} \).

Proof. Each of the \( \sigma_i \)'s is a \( \sigma_{a,b,c} \) defined by (5.1). Remembering (2.11), (2.18) and (3.7), it is easily seen that

\[
(5.3) \quad c(\sigma_{a,b,c}) = \begin{cases} 
4c(c+1) + 4(a-1)(2n-a+2) + 2b(2n-2a-b+2) & \text{if } a \geq 1, \\
4c(c+1) + 2(b-1)(2n+1-b) & \text{if } a = 0.
\end{cases}
\]

On the other hand, denoting by \( \mu_i \) the highest weight of \( \sigma_i \), we have the following identities:

\[
\begin{align*}
\mu_3 &= \mu_2 + (\varepsilon_1 + \varepsilon_{n+1}), \\
\mu_6 &= \mu_5 + (\varepsilon_1 + \varepsilon_{n+1}), \\
\mu_9 &= \mu_8 + (\varepsilon_1 + \varepsilon_{n+1}) + \mu_7 + 2(\varepsilon_1 + \varepsilon_{n+1}), \\
\mu_{11} &= \mu_{10} + (\varepsilon_1 + \varepsilon_{n+1}), \\
\mu_{14} &= \mu_{13} + (\varepsilon_1 + \varepsilon_{n+1}), \\
\mu_1 &= \mu_4 + \varepsilon_{r+s} + \varepsilon_{r+s+1}, \\
\mu_2 &= \mu_12 + \varepsilon_r + \varepsilon_{r+1} = \mu_{15} + \varepsilon_r + \varepsilon_{r+1} + \varepsilon_{r+s} + \varepsilon_{r+s+1}, \\
\mu_6 &= \mu_{11} + \varepsilon_{r+s} + \varepsilon_{r+s+1}, \\
\mu_3 &= \mu_{14} + \varepsilon_r + \varepsilon_{r+1}.
\end{align*}
\]

Then it becomes clear that it suffices to compare \( \sigma_1, \sigma_3, \sigma_6 \) and \( \sigma_9 \). By (5.3) we get after some calculation:

\[
(5.4) \quad \begin{cases} 
c(\sigma_1) = t(t+2) + 4r(2n+1-r) + 2s(2n-s) - 4rs, \\
c(\sigma_3) = c(\sigma_1) + 2t + 4r + 4s - 4n + 1, \\
c(\sigma_6) = c(\sigma_1) + 2t + 4r - 4n - 3, \\
c(\sigma_9) = c(\sigma_1) + 4t + 8r + 4s - 8n.
\end{cases}
\]

Observing that \( \sigma_6 \) and \( \sigma_9 \) (resp. \( \sigma_3 \)) do not occur in \( \tau_{r,s,t} \) when \( r = 0 \) (resp. \( s = 0 \)), we get the result. \( \square \)

6. Proof of Theorem 1.4

Recall from Theorem 3.2 that our wish is to calculate explicitly the possible \( M \)-representations \( \sigma_{\max} \) which determine the exact value of the bottom \( \alpha_i \) of the continuous spectrum of \( \Delta_i \) on \( H^n(\mathbb{H}) \). (In the sequel, we shall assume as usual that \( 0 \leq l \leq 2n \), since \( \alpha_l = \alpha_{4n-l} \) by Hodge duality.)

Because such a representation \( \sigma_{\max} \) will be an \( M \)-irreducible constituent of some \( K \)-irreducible factor \( \tau_{r,s,t} \) decomposing \( \gamma_l \) as in Theorem 4.8, one should in theory decompose completely each summand \( \tau_{r,s,t} \) of \( \gamma_l \) into \( M \)-irreducible factors \( \sigma \) by using Theorem 5.2, and then compare the resulting Casimir values \( c(\sigma) \). Actually, this enormous task can be considerably simplified by adopting the following strategy.

We are only interested in the representations \( \tau_{r,s,t} \in \mathcal{K}(\gamma_l) \) whose restriction to \( M \) can effectively contain (at least) one of the possible representations \( \sigma_{\max} \). For convenience, let us give the following
Definition 6.1. A representation \( \tau_{r,s,t} \in \hat{K}(\gamma) \) is called a spectrally maximal factor of \( \gamma \) if \( \tau_{r,s,t}|_{\Delta} \) contains (at least) one of the possible representations \( \sigma_{\max} \) with positive multiplicity. With notation of Section 3, this amounts to say that the lowest eigenvalue \( \alpha_1 \) of the continuous spectrum of \( \Delta \) is attained at least on some subspace of \( L^2_c(G, \tau_{r,s,t}) \).

Now, Proposition 5.3 tells us that a representation \( \sigma_{\max} \) is necessarily of the form \( \sigma_{i}^{r,s,t} \) with \( i = 1, 3, 6, 9 \), each of these being some \( \sigma_{a,b,c} \) (notation is as in Theorem 5.2).

By (5.3), it is clear that \( c(\sigma_{a,b,c}) \) is an increasing function of each variable \( a, b, c \). Thus the spectrally maximal factors of \( \gamma \) are to be sought among the \( \tau_{r,s,t} \)'s whose parameters \( r, s, t \) are the greater possible. On the other hand, in the decompositions of Theorem 4.8, the parameter \( t \) of the \( \tau_{r,s,t} \)'s is completely determined once a bidegree \( (p, q) \) (such that \( p + q = l \) and \( p \leq q \)) is fixed, namely \( t = q - p \). As regards the first two parameters \( r, s \), their range is also determined by the bidegree \( (p, q) \) and more precisely, by the decomposition of the corresponding representation \( R_{p,q} \) (or \( R'_{p,2n-q} \)) (see the definitions of \( I_{p,q} \) and \( I'_{p,2n-q} \) in (4.19) & (4.20)). Consequently, for each fixed bidegree \( (p, q) \), we can retain only the \( \tau_{r,s,q-p} \)'s for \( (r, s) \in I_{p,q} \) or \( (r, s) \in I'_{p,2n-q} \) such that the sum \( r + s \) is maximal.

The general ideas being explained, let us pass to the calculations.

Lemma 6.2. (1) Let \( 0 \leq p \leq q \leq n \). Among the pairs \( (r, s) \in I_{p,q} \), those for which the sum \( r + s \) is maximal are the \( (p - 2u, q - p + 2u) \) with \( 0 \leq u \leq \left[ \frac{p-1}{2} \right] \).

(2) Let \( 0 \leq p \leq n < q \leq 2n \). Among the pairs \( (r, s) \in I'_{p,2n-q} \), those for which the sum \( r + s \) is maximal are

\[
\begin{cases} 
(n - q + p, p - q) & \text{if } q - p \leq n, \\
(0, 2n - q + p) & \text{if } q - p \geq n.
\end{cases}
\]

Proof. We assume first that \( 0 \leq p \leq q \leq n \) and we recall that the set \( I_{p,q} \) is determined by Theorem 4.10. Let us begin with parameters \( (r, s) \) which correspond to summands \( \tau_{r,s} \) of the subrepresentation \( V_{p,q} \subset R_{p,q} \). We have here \( (r, s) = (j, q - p + 2(u - v)) \), with \( 0 \leq u \leq \left[ \frac{p-1}{2} \right], \ 0 \leq v \leq \left[ \frac{q-p-1}{2} \right] \) and \( 0 \leq j \leq p - 2u \). Thus

\[
r + s = j + q - p + 2(u - v) \leq p - 2u + q - p + 2u - 2v \leq q,
\]

with equality if and only if \( v = 0 \) and \( j = p - 2u \). Hence we obtain the pairs \( (r, s) \) of the form \( (p - 2u, q - p + 2u) \) with \( 0 \leq u \leq \left[ \frac{p-1}{2} \right] \).

Then it remains to examine the complementary part in \( R_{p,q} \) for each of the four cases listed in Theorem 4.10. For instance, when \( p \) and \( q \) are even, we see that the third term of \( R_{p,q} \) adds the pair \( (0, q) \) to our list, while the second term \( U_p \) can only contribute if \( p = q \) (the sums \( r + s \) are \( p \), which is itself \( \leq q \)) but in this case we just recover the pair \( (p, 0) = (p, q-p) \), already counted. The three other cases can be treated similarly, hence we have proved (1).

Next, assume that \( 0 \leq p \leq n < q \leq 2n \). Then \( R'_{p,2n-q} = R_{p,2n-q} \oplus V'_{p,2n-q} \) by Theorem 4.10, so that we can write

\[
I'_{p,2n-q} = I_{p,2n-q} \cup J_{p,2n-q},
\]

where

\[
J_{p,2n-q} = \{ (r, s) \in (0, \ldots, p) \times (0, \ldots, 2n - q) \text{ such that } \tau_{r,s} \subset V'_{p,2n-q} \}.
\]

Proceeding as in part (1), we see that the pairs \( (r, s) \in I_{p,2n-q} \) such that \( r + s \) is maximal are the \( (p - 2u, 2n - q - p - 2u) \) for \( 0 \leq u \leq \left[ \frac{p}{2} \right] \), in which case

\[
r + s = 2n - q.
\]
On the other hand, if \((r, s) \in I_{p,2n-q}\) then \((r, s) = (j-i, p+q-2u-2j)\) with \(0 \leq u \leq \left\lfloor \frac{p+1}{2} \right\rfloor\), 
\((n+p-q-2u)_+ \leq j \leq p - 1 - 2u\) and \(0 \leq i \leq j\). Consequently, 
\[
r + s = 2n + p - q - 2u - i - j
\]

\[(6.2)\]

\[
\leq 2n + p - q - (n + p - q)_+ = \begin{cases} 
  n & \text{if } q - p \leq n, \\
  2n - q + p & \text{if } q - p \geq n,
\end{cases}
\]

with equality when \(u = i = 0\) and \(j = (n + p - q)_+,\) i.e. when
\[
(r, s) = \begin{cases} 
  (n + p - q, q - p) & \text{if } q - p \leq n, \\
  (0, 2n - q + p) & \text{if } q - p \geq n.
\end{cases}
\]

Eventually, by comparing \((6.1)\) and \((6.2)\) we see that we can discard the pairs \((r, s) \in I_{p,2n-q}\) (because \(q \geq n + 1\)), and this proves \((2)\).

**Proposition 6.3.** If \(\tau_{r,s,t}\) is a spectrally maximal factor of \(\tau_l\), then \(\tau_{r,s,t}\) has to be a member of the following list:

\[
\begin{cases} 
  \tau_{k,l-2k,l-2k} & \text{for } 0 \leq k \leq \left\lfloor \frac{l}{2} \right\rfloor \\
  \tau_{l-n,l-2k,l-2k} & \text{for } 0 \leq k \leq \left\lfloor \frac{l-n}{2} \right\rfloor \\
  \tau_{l-n+k,l-2k,l-2k} & \text{for } \left\lfloor \frac{l-n}{2} \right\rfloor + 1 \leq k \leq l - n - 1 \\
  & \text{if } n + 1 \leq l \leq 2n,
\end{cases}
\]

Moreover, it occurs in \(\tau_l\) with multiplicity 1. (When \(n + 1 \leq l \leq 2n\), it will be understood that we discard the representations of the third type which overlap representations of the second type; this happens when \(l = n + 1\) or \(l = n + 2\).)

Notice that we do not assert that each \(K\)-type listed above is actually a spectrally maximal factor of \(\tau_l\), since this is not true in general, as we shall see at the end of this section. On the other hand, it is interesting to observe that the representations \(\tau_{k,l-2k,l-2k}\) (for \((l-n)_+ \leq k \leq \left\lfloor \frac{l}{2} \right\rfloor\)) are exactly the \(K\)-types which characterize hypereffective \(l\)-forms (see Proposition 4.12).

**Proof of Proposition 6.3.** Let us assume first that \(0 \leq l \leq n\). By Theorem 4.8 and the previous lemma, we see that the spectrally maximal factors can only be the following:

\[(p, q) = (0, l) \rightarrow \tau_{0,l,l}\]
\[(p, q) = (1, l - 1) \rightarrow \tau_{l-2,l-2}\]
\[(p, q) = (2, l - 2) \rightarrow \tau_{l-4,l-4}, \tau_{l-2,l-4}\]
\[(p, q) = (3, l - 3) \rightarrow \tau_{l-6,l-6}, \tau_{l-4,l-6}\]
\[(p, q) = (4, l - 4) \rightarrow \tau_{l-8,l-8}, \tau_{l-6,l-8}, \tau_{l-4,l-8}\]

\[
\vdots
\]

\[
[l \text{ even}] \quad (p, q) = \left( \frac{l}{2}, \frac{l+1}{2} \right) \rightarrow \tau_{\frac{l}{2},0,0}, \tau_{\frac{l}{2}-2,2,0}, \cdots, \tau_{0,\frac{l}{2},0}
\]
\[
[l \text{ odd}] \quad (p, q) = \left( \frac{l-1}{2}, \frac{l+1}{2} \right) \rightarrow \tau_{\frac{l-1}{2},1,1}, \tau_{\frac{l-1}{2}-2,3,0}, \cdots, \tau_{1,\frac{l-1}{2},1}
\]

But \(\tau_{0,l-2,l-4}\) cannot be a spectrally maximal factor, since we have also \(\tau_{0,l,l}\) in the list, which has greater second and third parameters. Similarly, we can discard \(\tau_{l-4,l-6}\) because of \(\tau_{l-2,l-2}\), and more generally we see that we can discard all terms except the first ones in each line of the previous list.
Consider now the case \( n + 1 \leq l \leq 2n \). As in the statement of Theorem 4.8 we must tackle differently the subcases \( p \leq q \leq n \) and \( p \leq n < q \), corresponding respectively to the sets \( I_{p,q} \) and \( I'_{p,2n-q} \). In the first case, we get the following list of candidates

\[
(p, q) = (l - n, n) \rightarrow \tau_{l-n,2n-l,2n-l}, \tau_{l-n,2n-l+2n-l}, \tau_{l-n,4,2n-l+4,2n-l}, \ldots
\]

\[
(p, q) = (l - n + 1, n - 1) \rightarrow \tau_{l-n+1,2n-l-2,2n-l-2}, \tau_{l-n-1,2n-l}, \tau_{l-n,2n-l-2}, \ldots
\]

\[
\vdots
\]

\( [l \text{ even}] (p, q) = (\lfloor \frac{l}{2} \rfloor, \frac{l}{2}) \rightarrow \tau_{\frac{l}{2},0,0}, \tau_{\frac{l}{2}-2,2,0}, \ldots, \tau_{0,\frac{l}{2},0}
\]

\( [l \text{ odd}] (p, q) = (\lfloor \frac{l-1}{2} \rfloor, \frac{l+1}{2}) \rightarrow \tau_{\frac{l-1}{2},1,1}, \tau_{\frac{l-1}{2}-2,3,0}, \ldots, \tau_{1,\frac{l-1}{2},1}
\]

and we argue as in the case \( 0 \leq l \leq n \) to see that the only possible spectrally maximal factors are the \( \tau_{k,l-2k,l-2k} \) for \( l - n \leq k \leq \lfloor \frac{l}{2} \rfloor \). Now, if \( p \leq n < q \), then \( q - p = l - 2p \) with \( 0 \leq p \leq l - n - 1 \), and \( q - p \geq n \) if and only if \( p \leq \lfloor \frac{l-n}{2} \rfloor \). By using the second part of the previous lemma, we obtain thus the following representations:

\[
(p, q) = (0, l) \rightarrow \tau_{0,2n-l,l}
\]

\[
(p, q) = (1, l-1) \rightarrow \tau_{0,2n-l+2,l-2}
\]

\[
\vdots
\]

\[
(p, q) = (\lfloor \frac{l-n}{2} \rfloor, l - \lfloor \frac{l-n}{2} \rfloor - 1) \rightarrow \tau_{0,2n-l+2\lfloor \frac{l-n}{2} \rfloor, l-2\lfloor \frac{l-n}{2} \rfloor}
\]

and

\[
(p, q) = (\lfloor \frac{l-n}{2} \rfloor + 1, l - \lfloor \frac{l-n}{2} \rfloor - 1) \rightarrow \tau_{n-l+2\lfloor \frac{l-n}{2} \rfloor + 1, l-2\lfloor \frac{l-n}{2} \rfloor + 1}
\]

\[
\vdots
\]

\[
(p, q) = (l - n - 1, n + 1) \rightarrow \tau_{l-n,2n-2l,2n-2l}
\]

The assertions concerning the multiplicities follow clearly from our discussion. \( \square \)

The previous result brings us very close to our final goal. It shows indeed that it remains only to compare the Casimir values of the \( M \)-irreducible factors of the representations listed. Moreover, Proposition 5.3 says that we can restrict to examine only a few of them in each case.

Assume first that \( 0 \leq l \leq n \). For short, let us set \( \sigma_{i,k} = \sigma_{i,k,l-2k,l-2k} \) for \( i = 1, 3, 6, 9 \) and \( k = 0, \ldots, \lfloor \frac{l}{2} \rfloor \) (we keep notation of Theorem 5.2). With (5.4) and some calculation we obtain

\[
\begin{align*}
c(\sigma_{1,k}) &= l(4n + 2 - l), \\
c(\sigma_{3,k}) &= l(4n + 2 - l) + 6l - 8k - 4n + 1 \quad [k \neq \frac{l}{2}], \\
c(\sigma_{6,l/2}) &= l(4n + 2 - l) + 2l - 4n - 3, \\
c(\sigma_{9,k}) &= l(4n + 2 - l) + 8(l - k - n) \quad [k \neq 0].
\end{align*}
\]

(The restrictions on \( k \) come from Proposition 5.3.) Consequently, we get

\[
\begin{align*}
c(\sigma_{\max}) &= \begin{cases} 
c(\sigma_{3,0}) &= l(4n + 2 - l) + 6l - 4n + 1 & \text{if } 1 \leq l \leq \lfloor \frac{4n-1}{6} \rfloor, \\
c(\sigma_{1,k}) &= l(4n + 2 - l) & \text{if } l = 0 \text{ or } \lfloor \frac{4n-1}{6} \rfloor + 1 \leq l \leq n - 1, \\
c(\sigma_{1,k}) &= n(3n + 2) & \text{if } l = n.
\end{cases}
\end{align*}
\]
Then Theorem 3.2 yields
\[
\alpha_l = \begin{cases} 
(2n - l)^2 + 8(n - l) & \text{if } 1 \leq l \leq \lfloor \frac{4n-1}{6} \rfloor, \\
(2n + 1 - l)^2 & \text{if } l = 0 \text{ or } \lfloor \frac{4n-1}{6} \rfloor + 1 \leq l \leq n.
\end{cases}
\]

Now suppose that \( n + 1 \leq l \leq 2n \). For \( i = 1, 3, 6, 9 \) we set
\[
\begin{align*}
\sigma_{i,k} &= \sigma_{i,k,l-2k,l-2k} \quad (l - n \leq k \leq \lfloor \frac{1}{2} \rfloor), \\
\sigma'_{i,k} &= \sigma_{i,0n-1+l,2k,l-2k} \quad (0 \leq k \leq \lfloor \frac{1-n}{2} \rfloor), \\
\sigma''_{i,k} &= \sigma_{i,n+2k-l,l-2k,l-2k} \quad (\lfloor \frac{-n}{2} \rfloor + 1 \leq k \leq l - n - 1; \ l \geq n + 3).
\end{align*}
\]
Formula (6.3) is still valid, and with the help of (5.4) we calculate likewise:
\[
(6.4) \quad \begin{cases}
c(\sigma'_{1,k}) = l(4n + 2 - l) - 4k(2n - 1 + l + k), \\
c(\sigma'_{3,k}) = l(4n + 2 - l) - 4k(2n - l + k) \quad [k \neq 0 \text{ if } l = 2n], \\
c(\sigma''_{1,k}) = c(\sigma''_{3,k}) = l(4n + 2 - l) + 4(n - l)(n + 1) - 4k(k - l - 1), \\
c(\sigma''_{9,k}) = l(4n + 2 - l) + 4(n - l)(n + 1) - 4k(k - l) + 2l + 1.
\end{cases}
\]
Comparing (6.3) and (6.4) we see easily that
\[
c(\sigma_{\max}) = \begin{cases} 
3l + n + 2(l - 1) + 4n - 2l + 1 & \text{if } n + 1 \leq l \leq 2n - 1, \\
2n(2n + 2) & \text{if } l = 2n,
\end{cases}
\]
and this gives
\[
\alpha_l = \begin{cases} 
(2n - l)^2 & \text{if } n + 1 \leq l \leq 2n - 1, \\
1 & \text{if } l = 2n,
\end{cases}
\]
achieving eventually the proof of the main assertion in Theorem 1.4. As concerns the fact that \( \alpha_l \) is always an eigenvalue attached to an hypereffective form, we observe that, in all cases, at least one \( \sigma_{\max} \) is some \( \sigma_{i,k} \) with \( (l - n)_{+} \leq k \leq \lfloor \frac{1}{2} \rfloor \). By definition, such a representation is an \( M \)-constituent of the spectrally maximal factor \( \tau_{k,l-2k,l-2k} \) which corresponds by Proposition 4.12 to a subspace of hypereffective \( l \)-forms and is multiplicity free in \( \eta \) by Proposition 6.3. Note however that the \( K \)-types of hypereffective forms do not represent all spectrally maximal factors in the case \( n + 1 \leq l \leq 2n \).

7. THE SPECTRUM OF THE BOCHNER LAPLACIAN

We devote this last section to the proof of Theorem 1.6. Let us denote respectively by \( B_l \) and \( B_{r,s,t} \) the restrictions of the Bochner Laplacian \( B = \nabla^* \nabla \) to \( C^\infty(G, \eta) \) and \( C^\infty(G, \tau_{r,s,t}) \) (here, \( \nabla \) denotes as usual the lift of the Levi-Civita connection to the differential form bundle). We have then the corresponding \( \texttt{Weitzenb"ock formulas} \) (use (3.6) and e.g. [BOS94], Proposition 3.1):
\[
\begin{align*}
\Delta_l &= B_l + \eta(\Omega_l), \\
\Delta_{r,s,t} &= B_{r,s,t} + \tau_{r,s,t}(\Omega_t),
\end{align*}
\]
where \( \Delta_{r,s,t} := -\Omega_{g_{C^\infty(G,\tau_{r,s,t})}} \) corresponds to the restriction of \( \Delta_l \) to \( C^\infty(G, \tau_{r,s,t}) \) when \( \tau_{r,s,t} \in \hat{K}(\eta) \). Since \( \tau_{r,s,t} \) is irreducible, the curvature term in (7.2) is scalar (hence the one in (7.1) is diagonal):
\[
\tau_{r,s,t}(\Omega_t) = -c(\tau_{r,s,t}) \text{Id},
\]
where the Casimir value of \( \tau_{r,s,t} \) is given by the formula
\[
c(\tau_{r,s,t}) = \langle \lambda_{r,s,t}, \lambda_{r,s,t} + 2\delta_t \rangle.
\]
An easy calculation using (2.10), (2.18) and (4.11) yields actually
\[(7.4)\quad c(\tau_{r,s,t}) = 2t(t + 2) + 4r(2n + 3 - r) + 2s(2n + 2 - s) - 4rs.\]

Now, assume that \(l = 2n\). Using Theorem 4.13 and (7.2), we see that each subspace \(H_k = L^2(G, \tau_{k,0,2n-2k})_\Delta\) of \(L^2(G, \tau_I)\) for \(0 \leq k \leq n\) is an eigenspace of \(B_I\), the corresponding (discrete) eigenvalue being
\[c(\tau_{k,0,2n-2k}) = 8(n - k)(n - k + 1) + 4k(2n + 3 - k) = 8n(n + 1) + 4k(k + 1 - 2n).\]

This proves the second part of Theorem 1.6.

Next, let \(\beta_l\) denote the infimum of the continuous spectrum of \(B_l\), for any \(l\). Gathering (7.1), (7.2), (7.3), Theorem 3.2 and Proposition 5.3, we see that
\[(7.5)\quad \beta_l = \min \{ \beta_{r,s,t} : (r, s, t)\text{ such that } \tau_{r,s,t} \in \hat{\chi}(\gamma) \},\]
where
\[\beta_{r,s,t} = (2n + 1)^2 + c(\tau_{r,s,t}) - \max_{i=1,3,6,9} c(\sigma_i^{r,s,t}).\]

When writing this formula, it is understood that the maximum is taken in fact over a subset of \(\{1, 3, 6, 9\}\) which depends on \((r, s, t)\) as in the statement of Proposition 5.3 (for instance, considering \(i = 6\) is useless when \(s \neq 0\)).

Define also
\[\gamma_i^{r,s,t} = c(\tau_{r,s,t}) - c(\sigma_i^{r,s,t}) \quad (i = 1, 3, 6, 9).\]

By (7.4) and (5.4) we get
\[\gamma_1^{r,s,t} = t(t + 2) + 8r + 4s,\]
\[\gamma_3^{r,s,t} = t^2 + 4r + 4n - 1,\]
\[\gamma_6^{r,s,t} = t^2 + 4r + 4s + 4n + 3,\]
\[\gamma_9^{r,s,t} = t(t - 2) + 8n.\]

These expressions are increasing functions of each variable \(r, s, t\). Because of (7.5), we thus look for the triples \((r, s, t)\) which are ‘minimal’ (in the sense that the corresponding \(\gamma_i^{r,s,t}\) are minimal) when \(\tau_{r,s,t}\) ranges through the set \(\hat{\chi}(\gamma)\). A clue for the solution to this problem is given by dealing first with the examples of low degree treated in Proposition 4.11. We obtain
\[(7.6)\quad (r, s, t)_{\min} = \begin{cases} (0, 1, 1) & \text{if } l = 1, 3, 5, \\ (0, 0, 2) \text{ or } (1, 0, 0) & \text{if } l = 2, 6, \\ (0, 0, 0) & \text{if } l = 4, \end{cases}\]

hence
\[\min_{r,s,t,i} \gamma_i^{r,s,t} = \begin{cases} 0^{0,1,1} = 12 & \text{if } l = 1, 3, 5, \\ \min(0^{0,0,2}, 0^{1,0,0}) = \gamma_1^{1,0,0} = 3 & \text{if } l = 2, 6, \\ 0 & \text{if } l = 4. \end{cases}\]

As a matter of fact, by using Theorem 4.8 and Theorem 4.10 it is easy to see that (7.6) and (7.7) do extend to any degree \(l\) by replacing the cases
a) \(l = 1, 3, 5\)
b) \(l = 2, 6\)
c) \(l = 4\)
respectively with

a') \( l \) odd
b') \( l \) even and not a multiple of 4
c') \( l \) a multiple of 4.

For instance, suppose that \( l = 4k \) for some \( k \in \mathbb{N} \). From (4.26) we see that \( R_{2k,2k} \) always contains the (trivial) representation \( \tau_{0,0} \). Hence, \( \tau_l \) always contains \( \tau_{0,0,0} \) by Theorem 4.8. The other verifications are similar and left to the reader.

As a consequence of this discussion, we have proved Theorem 1.6. Note that hyper-effective forms play obviously the same role for the spectral information of the Bochner Laplacian \( \mathcal{B}_l \) as they do in the case of the Hodge-de Rham Laplacian \( \Delta_l \).

Let us finish this article with an ultimate observation. One might be surprised by the result of Theorem 1.6, in the sense that the continuous spectrum of the Bochner Laplacian \( \mathcal{B}_l \) only depends on the congruence of \( l \) modulo 4. As we know, this differs from the case of real hyperbolic spaces \( H^n(\mathbb{R}) \), for which the curvature term \( \tau_l(\Omega_\mathbb{R}) \) is scalar, equal to \( -(n-l) \text{Id} \), so that Theorem 1.1 and Theorem 1.2 yield:

**Theorem 7.1.** Let \( \mathcal{B}_l \) denote the Bochner Laplacian on a real hyperbolic space \( H^n(\mathbb{R}) \).

1. The continuous \( L^2 \) spectrum of \( \mathcal{B}_l \) has the form \([\beta_l, +\infty)\), with
   \[
   \beta_l = \begin{cases} 
   \left(\frac{n-1}{2}\right)^2 + l & \text{if } l \leq \left[\frac{n-1}{2}\right], \\
   \frac{n^2+1}{4} & \text{if } n \text{ is even and } l = \frac{n}{2}, \\
   \left(\frac{n+1}{2}\right)^2 - l & \text{if } l \geq \left[\frac{n+1}{2}\right].
   \end{cases}
   \]

2. The discrete \( L^2 \) spectrum of \( \mathcal{B}_l \) is empty, unless \( n \) is even and \( l = \frac{n}{2} \), in which case it reduces to the sole eigenvalue \( \frac{n^2}{4} \).

However, we must not think that the result of Theorem 1.6 constitutes a sporadic example. Indeed, by using the results of [Ped99], one can prove (as above) the following statement.

**Theorem 7.2.** Let \( \mathcal{B}_l \) denote the Bochner Laplacian on a complex hyperbolic space \( H^n(\mathbb{C}) \).

1. The continuous \( L^2 \) spectrum of \( \mathcal{B}_l \) has the form \([\beta_l, +\infty)\), with
   \[
   \beta_l = \begin{cases} 
   n^2 & \text{if } l \text{ is even}, \\
   n^2 + 3 & \text{if } l \text{ is odd}.
   \end{cases}
   \]

2. The discrete \( L^2 \) spectrum of \( \mathcal{B}_l \) is empty, unless \( l = n \), in which case it consists of the \( n+1 \) eigenvalues \( 2n(n+1) + 4k(k-n) \), for \( 0 \leq k \leq n \).

**References**


Laboratoire de Mathématiques (UMR 6056), Université de Reims, Moulin de la Housse, B.P. 1039, 51687 Reims Cedex 2, France

E-mail address: emmanuel.pedon@univ-reims.fr